Modeling Particle Interactions in High-Energy Theory

PHY-855 Quantum Field Theory, SS23, PHY-959, FS23, Michigan State University

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Abstract

This review considers standard approaches for describing particle propagation and interaction. Quantized fields are introduced. A model of quantum electrodynamics is developed. The anomalous electron magnetic moment is determined at one-loop precision, illustrating common problem solving methods used with perturbative quantum field theories (pQFT). The predictive scope of pQFT is discussed. The Standard Model is then presented in analogy. These are a collection of notes combined from the courses PHY-855 and PHY-959 taught at Michigan State University.

Contents

1 Perturbative Quantum Field Theories

General quantized field theories begin with a classical system expressed through a suitable Lagrangian. This Lagrangian and related Hamiltonian may be considered in the continuum limit, where discrete coordinates and parameters are converted to reflect a field, $\phi(\vec{x}, t)$. The Lagrange density, $\mathscr{L}$, representing the integrand of the Lagrangian, along with the Euler Lagrange equation determines the equations of motion of the system.

A classical field is then promoted to an operator, quantizing the theory. Formalism describing particle states, commutation relations between field operators, conjugate momenta, and creation and annihilation operators are then introduced. The physical propagation of one particle between two positions, the expectation of the product of two fields, requires a dependency on causality. Consideration for the time ordering of fields is included in the standard descriptions of particle propagators.

The functional structure of a propagator reflects physical features of the theory. In the case of massive scalar particles, the position of a complex pole represents the physical mass. The residue relates to the field strength renormalization.

The expectation of time-ordered field products is the Green's function, $G(\vec{x}_1, ..., \vec{x}_n)$ of the process. Fourier transform of the position-space Green's function gives the momentum-space representation, $\tilde{G}(\vec{p}_1, ..., \vec{p}_n)$. In an interacting theory, j incoming and k outgoing states are represented with, $G(\vec{x}_1, ..., \vec{x}_j, \vec{y}_1, ..., \vec{y}_k)$.

Interactions between particles can be incorporated perturbatively. The Hamiltonian is expressed in the interaction picture and the interaction potential, $\hat{V}_I(t)$, is identified. Time evolution is then related to $\hat{V}_I(t)$. After individual fields are represented in the interaction picture, dependence of the Green's function on the potential is made clear. The expectation is then expanded perturbatively with respect to this argument. The truncated Green's function, $\tilde{G}_{trunc}(\vec{p}_1, ..., \vec{p}_n)$, removes dependence on external leg corrections. Evaluating the Green's function for propagators and interaction terms helps to identify Feynman rules for a theory. These rules can be mapped to a diagram of interest, and the Green's function evaluated. The process's probability can be further related to a cross section or decay rate.

Determining a Green's function at one-loop precision introduces an ultraviolet divergence related to the unbounded loop momentum. A regularization scheme is used to derive physical meaning for the theory parameters. Dimensional regularization is most common in high-energy physics. The dimension, d of the theory is left variable, where convergent manipulations can be performed. A set of renormalization conditions are introduced to absorb further divergences, redefining parameters of the theory. The MS, minimal subtraction, or modified minimal subtraction scheme, are standard in perturbative QCD (quantum chromodynamics). On-shell renormalization conditions are widely used in QED (quantum electrodynamics). Last, the regulator is removed or replaced with its legitimate value, leaving finite observables. In practice, an estimation to second or third-loop order is determined. The resulting level of error and applicability depends on the choice of renormalization.

1.1 Scalar Fields

The Lagrangian density for a real scalar field, $\phi(\vec{x}, t)$, with a kinetic and mass term is:

$$\mathscr{L} = \frac{1}{2}\big(\partial_\mu \phi\big)\big(\partial^\mu \phi\big) - \frac{1}{2}m^2\phi^2. \tag{1}$$

With the Euler Lagrange equation, motion of this non-interacting system is found as the Klein Gordon equation:

$$\big(\partial_\mu \partial^\mu + m^2\big)\phi(\vec{x}, t) = 0. \tag{2}$$

Canonical quantization is performed, and the field is exchanged for a quantum field operator, $\hat{\phi}(\vec{x}, t)$. Equal time commutators for the field and its conjugate momentum are given below:

$$[\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y}), \tag{3}$$

$$[\hat{\phi}(\vec{x}, t), \hat{\phi}(\vec{y}, t)] = 0, \tag{4}$$

$$[\hat{\pi}(\vec{x}, t)), \hat{\pi}(\vec{y}, t)] = 0. \tag{5}$$

These relations reflect the fundamental brackets found in the classical theory, and the uncertainty relation between observables. Taking the Fourier transform of $\hat{\phi}(\vec{x}, t)$,

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} 2\omega_p \bigg(\hat{a}(\vec{p})e^{-i(\omega_p t - \vec{x}\cdot\vec{p})} + \hat{a}^\dagger(\vec{p})e^{i(\omega_p t - \vec{x}\cdot\vec{p})}\bigg). \tag{6}$$

The operators, $\hat{a}(\vec{p})$ and $\hat{a}^\dagger$, act to remove and create individual particle states of a given momentum. The operators behave as:

$$[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p'})] = (2\pi)^3 2\omega_p \delta^{(3)}(\vec{p} - \vec{p'}), \tag{7}$$

$$[H, \hat{a}(\vec{p})] = -\omega_p \hat{a}(\vec{p}), \tag{8}$$

$$[H, \hat{a}^\dagger(\vec{p'})] = \omega_p \hat{a}^\dagger(\vec{p'}), \tag{9}$$

$$[\hat{a}^\dagger(\vec{p}), \hat{a}^\dagger(\vec{p})] = 0, \tag{10}$$

$$[\hat{a}(\vec{p}), \hat{a}(\vec{p})] = 0. \tag{11}$$

The expectation of time-ordered particle propagation between positions, $\langle 0| T(\hat{\phi}(\vec{x})\hat{\phi}(\vec{y})) |0\rangle = D_F(\vec{x})$. The causal solution for this scalar field propagator, the Feynman-Stuckelberg propagator, is:

$$D_F(\vec{x}) = \int \frac{d^3p}{(2\pi)^3 2E_p} \begin{cases} e^{-iE_pt}e^{i\vec{p}\cdot\vec{x}} & \text{t} > 0, \\ e^{iE_pt}e^{-i\vec{p}\cdot\vec{x}} & \text{t} < 0. \end{cases} \tag{12}$$

Prior to integration in the complex plane, the functional form of the integral can be related to,

$$D_F(\vec{x}) = \int \frac{d^4p}{(2\pi)^4} \frac{iZ}{p^2 - m^2_{phys} + i\epsilon}. \tag{13}$$

Here, m^2_{phys} is the physical particle mass, and pole in the complex plane. The field strength renormalization factor, Z, is the residue of the integrand. This expectation of field products is related to its Green's function, $G(\vec{x}_1, ..., \vec{x}_n)$, and momentum-space Green's function $\tilde{G}(\vec{p}_1, ..., \vec{p}_n)$.

More general processes with interactions are treated in the interaction picture of quantum mechanics. The free theory, $\hat{H}_o$, is separated from the interaction component, $\hat{H} = \hat{H}_o + \hat{H}_\lambda$. The time evolution operator is identified as, $\hat{U}(t_2, t_1) = e^{i\hat{H}^S_o \cdot (t_2 - t_0)} e^{i\hat{H}\cdot(t_2 - t_1)} e^{-i\hat{H}^S_o \cdot (t_1 - t_0)}$, where $\hat{H}^S_o$ represents the Schrodinger picture. Fields now take the form, $\phi_I(\vec{x}, t) = \hat{U}^\dagger(t, t_0)\phi(\vec{x}, t)\hat{U}(t, t_0)$. Expanding the original Green's function to reflect dependence on the interaction picture,

$$G(\vec{x}_1, ..., \vec{x}_n) = \frac{\langle 0| T(\hat{\phi}_I(\vec{x_1})...\hat{\phi}_I(\vec{x_n})\exp(i\int d^4x \mathscr{L}_I)) |0\rangle}{\langle 0| T(\exp(i\int d^4x \mathscr{L}_I)) |0\rangle}. \tag{14}$$

Here, $\mathscr{L}_I$ is the interaction term, related to the interaction potential, $\hat{V}_I = -\int d^3x \mathscr{L}_I$. This dependence then can be expanded perturbatively in the Green's function, introducing additional products of field operators. Contributions to the solution at a given order reflect process diagrams.

The truncated Green's function removes virtual contributions to each leg, identifying connected interactions,

$$\tilde{G}_{trunc}(\vec{p}_1, ..., \vec{p}_n, \vec{k}_1, ..., \vec{k}_m) = \frac{\tilde{G}(\vec{p}_1, ..., \vec{p}_n, \vec{k}_1, ..., \vec{k}_m)}{\prod_f \tilde{G}(\vec{p}_f) \prod_i \tilde{G}(\vec{k}_i)}. \tag{15}$$

Feynman rules related to components of each diagram, particle propagators and interaction vertices, are identified and used to represent additional graphical processes mathematically. Such rules can also be considered in momentum space.

1.2 Regularization and Renormalization Procedures

When evaluating higher-order diagrams, loops involving the interaction and reinteraction of particles introduce a dependence on virtual particle kinematics. As the momenta of these propagators are unbounded, the Green's function becomes divergent. In a single-loop integral,

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} = i \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{-(k_E^2 + m^2)} = -\frac{i}{(2\pi)^4} \Omega_4 \int_0^\infty d\kappa \frac{\kappa^3}{\kappa^2 + m^2}. \tag{16}$$

Here, the momentum vector is mapped from Minkowski space to Euclidean space with a Wick rotation. After integrating over the surface of a four-dimensional unit sphere, the result diverges with the Euclidean momentum normalization. General divergences related to unbounded particle momenta are handled with a regularization scheme and formal choice of renormalization conditions. These procedures allow for the definition of finite observables estimated with a perturbative expansion.

Dimensional regularization is used commonly in high-energy physics. Integrals divergent in four dimensions are expressed in terms of variable d dimensions. A Wick rotation is performed, followed by integration over the d-dimensional unit sphere, and radial integration. A convergent result is determined for some value of d. Later, this regulator is replaced by its true value, and any observables are left finite. The result of the integral above is related to,

$$\int d^dk \frac{1}{(k^2 - \triangle)^n} = (-1)^n i\pi^{d/2} \frac{\Gamma(n - d/2)}{\Gamma(n)} \triangle^{d/2-2}. \tag{17}$$

Further divergences appear from considering the one-particle irreducible contributions to the Green's function. These diagrams involve interaction vertices or virtual momenta which cannot be further segmented. In considering perturbative contributions to a propagating scalar particle, irreducible diagrams contribute to the Green's function with self-energy, $\Sigma(p)$. As repeated contributions to the self-energy can be made, connected by a propagator, the Dyson series for a connected (two-point interaction) Green's function follows:

$$\begin{aligned} \tilde{G}_c(\vec{p}) = G_c(\vec{x_1}, \vec{x_2}) &= D_F + D_F i\Sigma(p) D_F + D_F i\Sigma(p) D_F i\Sigma(p) D_F + \ldots \\ &= \frac{D_F}{1 - i\Sigma(p) D_F} = \frac{i}{i/D_F + \Sigma(p)} = \frac{i}{p^2 - m_o^2 + \Sigma(p) + i\epsilon}. \end{aligned} \tag{18}$$

This result for $\tilde{G}_c(\vec{p})$ is expressed as the vertex function for a given theory, $\Gamma(p) = i\tilde{G}_c^{-1}(\vec{p})$. Dependence on the propagator momentum, $\vec{p}$, can introduce divergences, requiring renormalization of the theory. Specifically, renormalization conditions are introduced to give physical meaning to the mass and field strength renormalization. In the on-shell scheme, the renormalized (two-point) vertex function is set as,

$$\Gamma_R^{(2)}|_{p^2=m^2_{phys}} = 0, \tag{19}$$

$$\frac{\partial}{d\partial^2}\Gamma_R^{(2)}|_{p^2=m^2_{phys}} = 1. \tag{20}$$

Alternately, the mininal subtraction (MS) scheme is designed to remove ultraviolet divergent terms, poles related to $1/\varepsilon$. In general, a factor of $s_\varepsilon = (4\pi)^\varepsilon e^{\gamma_E \cdot \varepsilon}$ is kept, where γ_E is the Euler-Mascheroni constant. Then, $\lambda_o s_\varepsilon = \lambda_{\overline{MS}} \mu_R^{2\varepsilon} Z_\lambda$. Here, the bare coupling of a Lagrangian, λ_o, is related to the renormalization coupling, $\lambda_{\overline{MS}}$, renormalization scale, μ_R, and coupling renormalization constant, Z_λ. The last is set such that Γ_R is finite in the zero-limit of ε. The coupling renormalization constant may then be expanded per-loop as,

$$Z_\lambda = 1 + \frac{1}{\varepsilon} Z_{\lambda,1}(\lambda_{\overline{MS}}) + \frac{1}{\varepsilon^2} Z_{\lambda,2}(\lambda_{\overline{MS}}) + \ldots \tag{21}$$

The Green's function for particle propagation or interaction is reevaluated, deriving new Feynman rules in terms of the renormalized theory parameters.

Finally, if a different renormalization scale, M is considered, the transformation of the vertex function reveals a relation between the dimensionality of field parameters – the Callan-Symanzik equation:

$$M \to e^s M, \tag{22}$$

$$\left(M\frac{\partial}{\partial M} + \beta(\lambda_R)\frac{\partial}{\partial \lambda_R} - n\gamma(\lambda_R) + m_R\gamma_M(\lambda_R)\frac{\partial}{\partial m_R}\right)\Gamma^{(n)} = 0. \tag{23}$$

Here, $\gamma(\lambda_R)$, $\beta(\lambda_R)$ and $\gamma_M(\lambda_R)$ are related to the anomalous dimension of the field, the beta function, and anomalous dimension of mass, as functions of renormalization parameters.

1.3 Dirac Fermion Fields

Fermions in a Dirac field are described by the the Lagrangian density,

$$\mathscr{L} = \overline{\psi}(i\partial_\mu\gamma^\mu - m)\psi. \tag{24}$$

To describe fermion states, the Weyl spinor representation of the Lorentz group is considered. Left and right-handed spinors transform as:

$$\phi_{R/L} = \begin{bmatrix} \vdots \end{bmatrix} \to \begin{bmatrix} \vdots \end{bmatrix} e^{i\frac{\vec{\tau}}{2}\vec{\theta} \pm \frac{\vec{\tau}}{2}\vec{\phi}}. \tag{25}$$

A Dirac spinor is then constructed, with Lorentz transformation properties,

$$\psi = \begin{bmatrix} \phi_R \\ \phi_L \end{bmatrix} \to \begin{bmatrix} e^{i\vec{\sigma}(\vec{\theta}-i\vec{\phi})/2} & 0 \\ 0 & e^{i\vec{\sigma}(\vec{\theta}+i\vec{\phi})/2} \end{bmatrix} \psi. \tag{26}$$

Additional transformation properties for parity should also be noted. Given a spinor field with an initial fermion at rest, transforming to a different reference frame reveals the equation of motion of the system, the Dirac equation,

$$(\gamma_\mu p^\mu - m)\psi(p) = 0. \tag{27}$$

Here, γ_μ is a matrix of Pauli matrices, σ_i with σ_0 treated as the identity,

$$\gamma^i = \begin{bmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{bmatrix}. \tag{28}$$

The product of matrices, $i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$. Additionally, these matrices correspond to a Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. A more compact notation expresses the contracted product of four-vectors with γ_μ, $\not{a} = a_\mu\gamma^\mu$. Solutions to the Euler Lagrange equation may be separated as,

$$(i\not{p} - m)\psi(p) = 0, \tag{29}$$

$$\overline{\psi}(p)(i\not{p} + m) = 0. \tag{30}$$

States are be expressed as, $\psi(x) = v(p)e^{ip\cdot x}$ and $\psi(x) = u(p)e^{-ip\cdot x}$, two separate solutions to the equation of motion, corresponding to a particle and anti-particle. In momentum space,

$$(\not{p} - m)u(p) = 0, \tag{31}$$

$$(\not{p} + m)v(p) = 0. \tag{32}$$

The theory can now be quantized. Expressing the field operator, $\hat{\psi}(x)$, in terms of creation and annihilation operators,

$$\hat{\psi}(x) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\Big(\hat{a}_s(p)u^{(s)}(p)e^{-i\vec{p}\cdot\vec{x}} + \hat{b}_s^\dagger(p)v^{(s)}(p)e^{i\vec{p}\cdot\vec{x}}\Big). \tag{33}$$

The summation is over spin, which is not discussed here in detail. Solutions for $\hat{a}_s(p)$ and $\hat{b}_s(p)$ can be found in terms of the field operator and momentum space solutions. The theory is quantized with anticommutators as below,

$$\Big\{\hat{\psi}(\vec{x},t), \hat{\pi}(\vec{y},t)\Big\} = i\delta^{(3)}(\vec{x}-\vec{y}), \tag{34}$$

$$\Big\{\hat{\psi}(\vec{x},t), \hat{\psi}(\vec{y},t)\Big\} = 0, \tag{35}$$

$$\{\hat{\pi}(\vec{x},t)), \hat{\pi}(\vec{y},t)\} = 0, \tag{36}$$

$$\Big\{\hat{a}_r(\vec{p}), \hat{a}_s^\dagger(\vec{p'})\Big\} = \Big\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p'})\Big\} = (2\pi)^3 2E_p\delta_{rs}\delta^{(3)}(\vec{p}-\vec{p}'), \tag{37}$$

$$\Big\{\hat{a}_r(\vec{p}), \hat{a}_s(\vec{p'})\Big\} = \Big\{\hat{b}_r(\vec{p}), \hat{b}_s(\vec{p'})\Big\} = 0, \tag{38}$$

$$\Big\{\hat{a}_r(\vec{p}), \hat{b}_s(\vec{p'})\Big\} = \Big\{\hat{a}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p'})\Big\} = 0. \tag{39}$$

Exchange of particles will follow Fermi statistics with these anticommutation relations.

Propagation of a free particle is again found by considering a time-ordered product of field operators,

$$\begin{aligned}\langle 0|\, T(\psi_a(x)\overline{\psi}_b(y))\, |0\rangle &= (i\partial\!\!\!/ + m)_{ab}\int \frac{d^4\vec{p}}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(\vec{x}-\vec{y})},\\ &= \int \frac{d^4\vec{p}}{(2\pi)^4}\frac{i(p\!\!\!/ + m)_{ab}}{p^2-m^2+i\epsilon}e^{-ip\cdot(\vec{x}-\vec{y})}.\end{aligned} \tag{40}$$

Additional Feynman rules are found by considering the ordered effect of the field operator on momentum states. Momentum-space Feynman rules are summarized for a fermion field below.

1. - Incoming fermions of momentum $\vec{p}$, spin s, correspond to a factor of $u^{(s)}(\vec{p})$,
   - Incoming antifermions correspond to a factor of $\overline{v}^{(s)}(\vec{p})$,
   - Outgoing fermions correspond to a factor of $\overline{u}^{(s)}(\vec{p})$,
   - Outgoing antifermions correspond to a factor of $v^{(s)}(\vec{p})$.
   - Fermion propagators receive a factor,

   $$\frac{i(p\!\!\!/ + m)}{p^2-m^2+i\epsilon}. \tag{41}$$

2. - Closed fermion loops require a change of sign.
   - The exchange of external lines leads to an additional change of sign.
   - Spinor indices may be optionally ignored by constructing the Green's function against the direction of the arrow. This result is the trace of the matrix product.
   - Symmetry factors of the diagram must be accounted for.

3. Unknown four-momenta are integrated over, corresponding to the integration of the original expansion of the interaction Lagrangian density,

$$\int \frac{d^4k}{(2\pi)^4}\Big(\ \dots\ \Big). \tag{42}$$

1.4 Quantum Electrodynamics

In quantum electrodynamics, interactions between both fermions and bosons (photons) are described. Again, a classical description of the photon will be quantized, and rules describing propagation and interaction vertices identified.

The electric and magnetic field relations to scalar and vector potentials, $\phi, \vec{A}$ are stated,

$$\vec{B} = \vec{\nabla} \times \vec{A}, \tag{43}$$

$$\vec{E} = \partial_t \vec{A} - \vec{\nabla}\phi, \tag{44}$$

$$A^\mu = (\phi, \vec{A}). \tag{45}$$

The field strength tensor, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, is then used to form a Lorentz-invariant product in the classical Lagrangian density,

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{46}$$

Applying the Euler Lagrange equation, the solution is the set of inhomogeneous equations, $\partial_\mu F^{\mu\nu} = 0$. Fourier decomposition of A^μ expresses the creation and annihilation operators, as well as the polarization, $\varepsilon_\mu^{(\lambda)}(k)$,

$$A^\mu(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3 2E_k} \sum_{\lambda=0} \varepsilon_\mu^{(\lambda)}(k) \left[a_\lambda(k) e^{-i\vec{k}\cdot\vec{x}} + a_\lambda^*(k) e^{i\vec{k}\cdot\vec{x}} \right]. \tag{47}$$

The vector photon field and related operators are quantized as,

$$[\hat{A}_\mu(\vec{x}, t), \hat{\pi}_\nu(\vec{y}, t)] = i g_{\mu\nu} \delta^{(3)}(\vec{x} - \vec{y}), \tag{48}$$

$$[\hat{A}_\mu(\vec{x}, t), \hat{A}_\nu(\vec{y}, t)] = 0, \tag{49}$$

$$[\hat{\pi}_\mu(\vec{x}, t)), \hat{\pi}_\nu(\vec{y}, t)] = 0, \tag{50}$$

$$[a_\lambda(k), a^\dagger_{\lambda'}(k')] = -g^{\lambda\lambda'} 2k_o (2\pi)^3 g^{(3)}(\vec{k} - \vec{k}'). \tag{51}$$

An additional gauge fixing term is added to the Lagrangian to preserve covariance,

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2f} (\partial_\mu A^\mu)^2. \tag{52}$$

Here, f is a gauge fixing scale. In the choice of the Feynman gauge, $f = 1$. Considering again the Dirac fermion field, the representative Lagrangian density must also be invariant under general gauge transformations. This is accomplished with use of the covariant derivative, $D_\mu = \partial_\mu - ieA_\mu(x)$. With the combination of electron (fermion) and photon fields, the full Lagrangian density for quantum electrodynamics can be considered,

$$\mathscr{L}_{QED} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2f} (\partial_\mu A^\mu)^2. \tag{53}$$

Additional momentum-space Feynman rules can be determined for the photon, to be used in combination with the Dirac fermion field:

1. • The propagator for the photon is represented as,

$$\frac{-i(g_{\mu\nu} - (1 - f) p_\mu p_\nu / p^2)}{p^2 + i\partial}. \tag{54}$$

- The interaction vertex between two electrons (fermions) and one photon is represented by the strength of the coupling, $ie\gamma^{\mu}$.
- Incoming photons are represented by their polarization, $\varepsilon^{\mu}(p)$,
- Outgoing photons are represented as $\varepsilon^{*\mu}(p)$.

2.
- Reflecting summation over polarization, $\sum_{pol.} u(p)\overline{u}(p) = \not{p} + m$,
- $\sum_{pol.} v(p)\overline{v}(p) = \not{p} - m$,
- $\sum_{pol.} \varepsilon^{\mu}(p)\varepsilon^{\nu *}(p) = -g^{\mu\nu}$.

1.4.1 Electron Magnetic Moment with Projector

One application of perturbative QED is the predicted value of the anomalous electron magnetic moment, g. An estimate can be found at one-loop precision, demonstrating the evaluation of a diagram with standard integration and tensor decomposition methods. An approach involving the construction of a projector is discussed in this section [2]. In the following section, tensor integral reduction with Passarino-Veltman cofficients is considered. The software Mathematica is used for symbolic evaluation. The package FeynCalc supports objects specific to pQFT.

In this prediction, dimensional regularization is used. A number of relations are utilized to remove operators or contract products. In addition to earlier-stated relations between γ_{μ} matrices,

$$\not{p}u(p) = mu(p), \tag{55}$$

$$\not{p}v(p) = -mv(p), \tag{56}$$

$$\not{p}^2 = p^2, \tag{57}$$

$$\gamma_{\lambda}\gamma^{\lambda} = d. \tag{58}$$

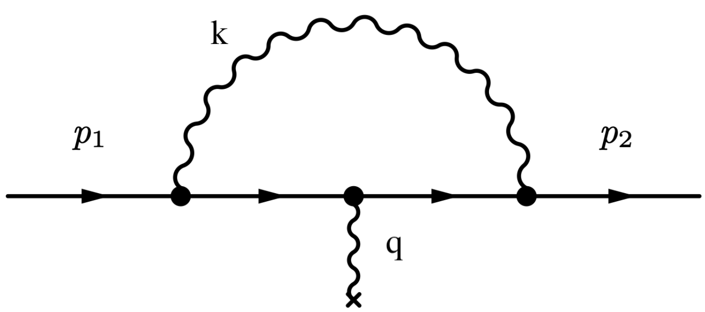

Figure 1: "One-Loop Interaction Vertex Correction": Pictured is the one-loop contribution to the interaction vertex with labeled momenta. A diagram was taken from [5] and relabeled to create this image.

Considering the interaction vertex between two electrons and an incoming photon, the difference in momentum of an incoming and outgoing electron are equivalent to the incident photon energy, $\vec{q} = \vec{p}_1 - \vec{p}_2$. At one-loop order, the interaction involves an internal loop photon of momentum, $\vec{k}$. A general element of the four-current can be expressed in terms of form factors as:

$$\begin{aligned} \langle \alpha_f | \, J_{\mu}(x) \, |\alpha_i\rangle = \overline{u}_f(p_2)\Bigg[&F_1\left(\frac{q^2}{m^2}\right)\gamma_{\mu} - \frac{i}{2m}F_2\left(\frac{q^2}{m^2}\right)\sigma_{\mu\nu}q^{\nu} + \frac{1}{m}F_3\left(\frac{q^2}{m^2}\right)q_{\mu} + \\ &\gamma_5\left(G_1\left(\frac{q^2}{m^2}\right)\gamma_{\mu} - \frac{i}{2m}G_2\left(\frac{q^2}{m^2}\right)\sigma_{\mu\nu}q^{\nu} + \frac{1}{m}G_3\left(\frac{q^2}{m^2}\right)q_{\mu}\right)\Bigg]u_i(p_1)e^{i\vec{q}\cdot\vec{x}}. \end{aligned} \tag{59}$$

The sum of electron momenta can be considered constant, held at $P = (\vec{p}_1 + \vec{p}_2)/2$. Simplifying further, the current can be related to the vertex function,

$$\overline{u}_f(p_2)\Gamma^\mu(p_1, p_2, q)u_i(p_1) = \overline{u}_f(p_2)\left(F_1\left(\frac{q^2}{m^2}\right)\gamma^\mu + \frac{i}{2m}F_2\left(\frac{q^2}{m^2}\right)\sigma^{\mu\nu}q_\nu\right)u_i(p_1). \tag{60}$$

Here, $F_1(q^2/m^2)$ reflects charge, while $F_2(q^2/m^2)$ is the magnetic form factor. The anomalous magnetic moment of a fermion is found by extracting,

$$a \equiv \frac{1}{2}(g-2) = F_2(0). \tag{61}$$

This value can be determined by projecting out the value of $F(q^2/m^2)$. Such a projector would be formed as a linear combination of vectors spanning the original expression [2],

$$N_\mu = (\not{p}_1 + m)\left[g_1\gamma_\mu - \frac{1}{m}g_2 P_\mu - \frac{1}{m}g_3 q_\mu\right](\not{p}_2 + m). \tag{62}$$

Taking the trace, $\mathrm{Tr}(N_\mu \Gamma^\mu)$, leads to a system equations determining the values g_1, g_2 and g_3. Ultimately, an expression for the functional form factor, $F_2(q^2/m^2)$ is derived. The value of $F_2(0)$ can also be projected out directly. Expressing the vertex function as an expansion in q_μ [2],

$$\Gamma^\mu(P, q) \approx \Gamma^\mu(P, 0) + q_\nu \frac{\partial}{\partial_\nu}\Gamma^\mu(P, q)|_{q=0} \equiv V^\mu(P) - q_\nu T^{\mu\nu}(p). \tag{63}$$

The tree level interaction vertex is, $\Gamma^\mu_0 = \gamma^\mu$. At one-loop order, the first correction to the interaction is,

$$J^\mu_1 = \overline{u}_f(p_2)\left[\int \frac{d^d k}{(2\pi)^d}\frac{(ie\gamma^\nu)i(\not{p}_2 + \not{k} + m)}{(p_2 + k)^2 - m^2 + i\epsilon}\frac{(ie\gamma^\mu)i(\not{p}_1 + \not{k} + m)}{(p_1 + k)^2 - m^2 + i\epsilon}\frac{(ie\gamma^\ell)(-i)g^{\nu\ell}}{k^2 + i\epsilon}\right]u_i(p_1). \tag{64}$$

To express as a function of $\vec{q}$, $\vec{p}_1$ and $\vec{p}_2$ may be written as $\vec{p}_2 = (2\vec{P} - \vec{q})/2$, $\vec{p}_1 = (2\vec{P} + \vec{q})/2$. The choice of sign for the incoming photon momentum follows from [2]. Additionally, as translations of the virtual loop momentum do not impact the integral result, shifting $-\vec{k} \to -\vec{k} + \vec{P}$ will simplify vector operations in the numerator. After performing straightforward contractions, the new integral is expressed as (leaving complex adjustments implicit),

$$J^\mu_1 = \overline{u}_f(p_2)\left[e^3 \int \frac{d^d k}{(2\pi)^d}\frac{(\gamma^\nu)(-\not{q}/2 + \not{k} + m)}{(\vec{k} - \vec{q}/2)^2 - m^2 + i\epsilon}\frac{(\gamma^\mu)(\not{q}/2 + \not{k} + m)}{(\vec{k} + \vec{q}/2)^2 - m^2 + \epsilon}\frac{(\gamma^\nu)}{(\vec{P} - \vec{k})^2}\right]u_i(p_1). \tag{65}$$

After expanding the numerator product, vector and γ_μ matrix relations can be used to simplify each term. Noting $\partial_{q_\nu}\not{q} = \gamma^\nu$, the derivative of the integrand of J^μ_1 gives,

$$\begin{aligned} T^{\mu\nu} = -\frac{\partial}{\partial_\nu}\Gamma^\mu(P, q)|_{q=0} = \frac{1}{2\big(k^2 - m^2\big)\big(\vec{P} - \vec{k}\big)^2}\big(&ide^2 m[\gamma^\nu, \gamma^\mu] - ide^2\gamma^\nu\gamma^\mu(\gamma\cdot k) \\ &+ ide^2(\gamma\cdot k)\gamma^\mu\gamma^\nu + 4ie^2 m[\gamma^\mu, \gamma^\nu] + 6ie^2\gamma^\nu\gamma^\mu(\gamma\cdot k) - 6ie^2(\gamma\cdot k)\gamma^\mu\gamma^\nu\big). \end{aligned} \tag{66}$$

If the average spatial orientation of $\vec{k}$ is accounted for, the value of a is projected out as,

$$\begin{gathered} a = \frac{1}{2(d-1)(d-2)m^2}\times \\ \mathrm{Tr}\left(\frac{d-2}{2}[m^2\gamma_\mu - dp_\mu\not{p} - (d-1)mp_\mu]V^\mu + \frac{m}{4}(\not{p} + m)[\gamma_\nu, \gamma_\mu](\not{p} + m)T^{\mu\nu}\right). \end{gathered} \tag{67}$$

The numerator of the result can be expanded into a sum of vector products. The full sum is omitted here, but available in the attached Mathematica notebook, Projector.nb.

$$a = \frac{1}{(1-d)(2-d)\left(k^2-m^2\right)\left(\vec{P}-\vec{k}\right)^2}\left(2id^3e^2kP - \ldots + 16ie^2m^2 - 24ie^2kP\right). \tag{68}$$

The result is then refactored in terms of denominator components, $D_1 = (\vec{P}-\vec{k})^2$, and $D_2 = \left(k^2-m^2\right)$. This combination depends only on the scalar dimensionality, d, coupling, e^2, and mass, m^2. Vector dependencies are represented only in D_1 and D_2,

$$\begin{aligned} a = &-\frac{i(d-2)de^2D_1}{2(d-1)m^2D_2^2} - \frac{i(d-2)de^2}{2(d-1)m^2D_1} + \frac{2i(2d-3)e^2}{(d-1)D_2^2} + \\ &\frac{i(d-2)de^2}{(d-1)m^2D_2} - \frac{2ie^2}{D_1D_2}. \end{aligned} \tag{69}$$

Using integration-by-parts reduction, and application of the relations between γ_μ matrices, these integral forms can be related as,

$$\int d^dk\frac{1}{D_1} = 0, \tag{70}$$

$$\int d^dk\frac{1}{D_2^2} = \frac{(d-2)}{m^2}\int d^dk\frac{1}{D_2}. \tag{71}$$

An additional integral is rewritten after tensor decomposition,

$$\int d^dk\frac{D_1}{D_2^2} = \int d^dk\frac{1}{D_2} + 2m^2\int d^dk\frac{1}{D_2^2}. \tag{72}$$

The value of a can then be evaluated in terms of two scalar integrals,

$$a = -\frac{i(d-6)(d-2)e^2}{2m^2D_2} - \frac{2ie^2}{D_1D_2}. \tag{73}$$

The general solution for the tadpole-diagram integral is [1],

$$I^{d=4-2\epsilon}_{\text{tadpole}}(m^2) = \frac{\mu^{4-d}}{i\pi^{d/2}r_\Gamma}\int d^Dk\frac{1}{(k^2-m^2+i\varepsilon)} = m^2\left(\frac{\mu^2}{m^2-i\varepsilon}\right)^\epsilon\left[\frac{1}{\epsilon}+1\right] + \mathcal{O}(\epsilon). \tag{74}$$

While the bubble-diagram integral [1],

$$\begin{aligned} I_{\text{bubble}}(q^2,m_1^2,m_2^2) &= \frac{\mu^{4-d}}{i\pi^{d/2}r_\Gamma}\int d^Dk\frac{1}{(k^2-m_1^2+i\varepsilon)((k+q)^2-m_2^2+i\varepsilon)} \\ &= \mu^{2\epsilon}\left[\frac{1}{\epsilon} - \int_0^1 da\ln\left(-a(1-a)q^2 + am_2^2 + (1-a)m_1^2 - i\varepsilon\right)\right] + \mathcal{O}(\epsilon). \end{aligned} \tag{75}$$

Here, the dimensionality is expressed as $d = 4-2\epsilon$. After solutions to the relevant integrals are inserted, d is replaced by its value of four in the zero limit of ϵ. After dividing out remaining factors, the expression for a is,

$$a = \frac{\alpha}{2\pi} + \mathcal{O}(\epsilon). \tag{76}$$

Here, α is the fine structure constant.

1.4.2 Electron Magnetic Moment with Passarino-Veltman Reduction

In an alternate approach, $\overline{u}_f(p_2)\Gamma^\mu(p_1,p_2,q)u_i(p_1)$ can be expanded as a linear combination of tensor integrals. Passarino-Veltman tensor decomposition is then performed, expressed in terms of only $g^{\mu\nu}$, external momenta, and form factors dependent on Lorentz invariant quantities [3,4]. This set of factors, or Passarino-Veltman coefficients, map to a family of scalar integrals. Using these integrals as a basis for decomposition is standard in high-energy and the study of similar systems.

As an example, common one-loop integral forms can be reduced as,

$$\int d^4k \frac{k^\mu}{(k^2-m_1^2)((k+p)^2-m_2^2)} = p^\mu B_1(p^2,m_1^2,m_2^2), \tag{77}$$

$$\int d^4k \frac{k^\mu k^\nu}{(k^2-m_1^2)((k+p)^2-m_2^2)} = g^{\mu\nu}B_{00}(p^2,m_1^2,m_2^2) + p^\mu p^\nu B_{11}(p^2,m_1^2,m_2^2). \tag{78}$$

Additional propagators contributing to the loop are expressed in terms of additional coefficients. Multi-loop integrals can be reduced through similar methods with the assistance of integration-by-parts reductions.

In the case of the electron magnetic moment, the one-loop correction to the vertex function is expanded, and known kinematic relations are applied. To determine $F_2(q=0)$, the momentum transfer is assumed zero, and $q^2=0$. Tensor integrals are then decomposed in terms of their related set of scalar functions. The majority of this approach is worked explicitly elsewhere [5]. Evaluation is performed with FeynCalc, and illustrated in the notebook, Passarino_Veltman.nb. As only the component of Γ^μ proportional to q_ν is related to $F_2(0)$, further terms are dropped,

$$\begin{aligned}\Gamma^\mu = -2i\pi^2e^3m(p_1^\mu+p_2^\mu)\big(2C_1(m^2,0,m^2,0,m^2,m^2)+dC_{11}(m^2,0,m^2,0,m^2,m^2)-\\ 2C_{11}(m^2,0,m^2,0,m^2,m^2)+dC_{12}(m^2,0,m^2,0,m^2,m^2)-2C_{12}(m^2,0,m^2,0,m^2,m^2)\big).\end{aligned} \tag{79}$$

Values of the coefficients are then evaluated for their specific arguments, and the dimension is fixed to four,

$$\Gamma^\mu = -2i\pi^2e^3m\left(\frac{3}{32\pi^4m^2}-\frac{d}{g4\pi^4m^2}\right). \tag{80}$$

After removing an overall factor,

$$F_2(0) = \frac{\alpha}{2\pi}. \tag{81}$$

2 The Standard Model Lagrangian

The Standard Model (SM) is a minimal working representation of (almost) all known free parameters and symmetries governing the observed physical universe.

The Lagrangian satisfies three group symmetries:

$$SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{82}$$

The special unitary group, $SU(3)_C$, describes the space of matrix transformations which preserve the unit normalization of three vectors. This symmetry describes the color space of the model – specifically the color mediated in strong force interactions. Only colorless particles are observed, this concept of color confinement maintained with this symmetry.

The special unitary group, $SU(2)_L$ and unitary group $U(1)_Y$ describe other symmetries which do not involve color – left-handed Fermi interactions and hypercharge in the electroweak theory. The $SU(2)_L$ group represents the existence of left-handed fermion doublets (right-handed anti-fermion doublets). These particles are observed with nonzero, weak isospin, forming pairs. Right-handed fermions have no weak isospin, and exist individually in singlets. These relations pair generation members of charged leptons and neutrinos, and generation members of quarks. The hypercharge quantum number is then determined from this projected isospin and the electromagnetic charge via the Gell-Mann–Nishijima formula, $Y = 2(Q - I_3)$.

	I	II	III	I_3	Y	Q
Leptons	$\begin{pmatrix}\nu_e \\ e\end{pmatrix}_L$	$\begin{pmatrix}\nu_\mu \\ \mu\end{pmatrix}_L$	$\begin{pmatrix}\nu_\tau \\ \tau\end{pmatrix}_L$	$+1/2$ $-1/2$	-1 -1	0 -1
	e_R	μ_R	τ_R	0	-2	-1
Quarks	$\begin{pmatrix}u \\ d\end{pmatrix}_L$	$\begin{pmatrix}c \\ s\end{pmatrix}_L$	$\begin{pmatrix}t \\ b\end{pmatrix}_L$	$+1/2$ $-1/2$	$+1/3$ $+1/3$	$+2/3$ $-1/3$
	u_R	c_R	t_R	0	$+4/3$	$+2/3$
	d_R	s_R	b_R	0	$-2/3$	$-1/3$

Figure 2: Observed charge, weak isospin and hypercharge for quarks and leptons [**?**].

The two special unitary symmetries introduce 8 and 3 force carriers to the theory, following the dimensionality of generators introduced with $SU(3)$ and $SU(2)$. These first eight bosons represent the basis of color combinations which can be taken with a gluon. The following three bosons, W_1, W_2 and W_3 represent weak force carriers. Last, an additional neutral boson is introduced from the hypercharge symmetry, B.

The first two weak carriers rotate together to form the observed W^+, W^- mass eigenstates,

$$W^+ = \frac{W_1 - iW_2}{\sqrt{2}},\ W^- = \frac{W_1 + iW_2}{\sqrt{2}}. \tag{83}$$

The third weak carrier and neutral boson compose the photon and Z mass eigenstates:

$$\begin{bmatrix}\gamma \\ Z\end{bmatrix} = \begin{bmatrix}\cos(\theta_W) & \sin(\theta_W) \\ -\sin(\theta_W) & \cos(\theta_W)\end{bmatrix}\begin{bmatrix}B \\ W_3\end{bmatrix}. \tag{84}$$

Here, θ_W is the Weinberg mixing angle.

Last, a final mechanism is needed to set the masses of particles. This symmetry breaking occurs through a phase shift in the Goldstone boson, ϕ, representing the vacuum.

$$\phi = \begin{bmatrix}(V + i\phi^o)/\sqrt{2} \\ \phi^o/\sqrt{2}\end{bmatrix} \rightarrow \begin{bmatrix}(V + h)/\sqrt{2} \\ 0\end{bmatrix} \tag{85}$$

Here, V is the vacuum expectation, and h is the Higgs expectation. Three free parameters, ϕ^o, ϕ^+ and ϕ^- can be considered. When symmetry is spontaneously broken, the parameter ϕ^o settles to a specific value, and the freedom of one Goldstone boson is lost. The action of the covariant derivative on this field,

$$D_\mu = \partial_\mu + ig_2 W_\mu^a \tau^a, \tag{86}$$

$$(D_\mu\phi)(D_\mu\phi) \rightarrow \left(\frac{g_2 V}{\sqrt{2}}\right)^2 W^+W^- = M_W^2 W^+W^-. \tag{87}$$

Here, g_2 is the gauge coupling, W_μ^a represents the weak boson fields, and τ^a is the set of Pauli matrices. The now specified Higgs boson sets the mass of the W bosons, M_W. A similar constraint sets the mass of the Z boson. The specification of the three parameters, ϕ^o, ϕ^+ and ϕ^-, removes longitudinal freedoms from the Goldstone field, and imparts masses to the three weak bosons. In this way, the Higgs mass, W^+, W^- and Z masses are specified, while no such symmetry breaking is present to set the masses of the gluon and photon.

In summary, the fermionic elementary particles, leptons and quarks, are observed to obey a number of symmetries in nature. Describing these symmetries and explicit freedoms introduces a number of communicative forces represented by bosons. As in QED, the full Lagrangian can be written to express this content. We can first consider the covariant derivative which is used to express the kinetic terms of this system:

$$D_\mu = \partial_\mu - ig_1 \frac{Y}{2} B_\mu - ig_2 \frac{\tau^j}{2} W^j_\mu - ig_3 \frac{\lambda^a}{2} G^a_\mu. \tag{88}$$

- The second term acts on all fields of hypercharge, including left-handed fermionic doublets, right-handed singlets, and bosons. g_1 is the gauge coupling, Y is the hypercharge quantum number and unit matrix used for the symmetry group $U(1)$, and B_μ is the vector potential of the neutral boson.

- The third term represents the action of the weak force, acting only on $SU(2)$ doublets. g_2 is the gauge coupling, τ^j are the Pauli matrices used as generators for $SU(2)$, and W^j_μ are the vector potentials of the three weak bosons.

- The last term represents the action of colored fermions, quarks, following $SU(3)$. g_3 is the gauge coupling, λ^a are the eight Gell-Mann matrices spanning the space of $SU(3)$, and G^a_μ are the vector potentials of the eight possible gluon color combinations.

The general Lagrangian will consist of kinetic terms for both fermions and bosons, $\mathscr{L}_{f,kin}$, $\mathscr{L}_{b,kin}$, interaction terms for electroweak and QCD processes, $\mathscr{L}_{f,int}$, terms for massive particle, $\mathscr{L}_{mass}$, and the energy associated with scalar fields, $\mathscr{L}_\phi$:

$$\mathscr{L}_{SM} = \mathscr{L}_{f,kin} + \mathscr{L}_{b,kin} + \mathscr{L}_{mass} + \mathscr{L}_{f,int} + \mathscr{L}_\phi. \tag{89}$$

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