The Redshift - Luminosity Distance Relation

The best-known way to trace the evolution of the universe observationally is to look into the redshift - luminosity distance relation [1, 2]. The well-measured quantity of a far distant object is the redshift of light it emitted due to the expansion of the universe. The redshift z is related to the scale factor a by

$$\frac{\lambda_0}{\lambda} \equiv 1 + z = \frac{a_0}{a}.$$

From now on, the quantity with the subscript 0 means the value at present. Another important observational quantity is the distance to the object. There are several ways of measuring distances in the expanding universe. The luminosity distance d_L is defined by the relation

$$d_L^2 \equiv \frac{L}{4\pi F},$$

where L is the luminosity of the object and F is the measured flux from the object. For the object whose luminosity is know in some way, we can determine its luminosity distance from the measured flux.

What you will do in this project is to derive the relation between the redshift and the luminosity distance in a few cosmological models and compare it with the data obtained from the observations of type Ia supernovae.

The expanding universe is described by the FRW metic

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right],$$

where $K = 0, \pm 1$ depending on the spatial curvature of the universe.

(1) Show that the measured flux at the origin from the object of luminosity L located at $r = r_1$ is given by

$$F = \frac{L}{4\pi(a_0 r_1)^2(1+z)^2},$$

thus the luminosity distance to the object is $d_L = a_0 r_1 (1 + z)$. Consider why we have two factors of $(1 + z)$ in the numerator.

(2) r_1 is a function of the time t at which the light we see today was emitted by the object. From the fact that the light travels satisfying $ds^2 = 0$, derive

$$r_1 = f_K(z) \equiv \left(\begin{array}{ll} \sin f(z), & \text{for } K = +1, \\ f(z), & \text{for } K = 0, \\ \sinh f(z), & \text{for } K = -1, \end{array}\right.$$

where

$$f(z) = \frac{1}{a_0 H_0} \int_0^z \frac{dz'}{h(z')},$$

with the Hubble parameter $H = \dot{a}/a$ and $h(z) = H(z)/H_0$.

(3) The scale factor $a(t)$ satisfies the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3M_P^2} \sum_i \rho_i,$$

where ρ_i is the energy density of each component that fills the universe. Assume that the i-th component has the the equation of state $p_i = w_i \rho_i$ where w_i is a constant. When $w_i = 1/3,\ 0,\ -1$, it is called Radiation($i = R$), Matter($i = M$), and Cosmological Constant($i = \Lambda$), respectively. Then the energy density evolves as

$$\rho_i = \rho_{i0} \left(\frac{a}{a_0}\right)^{-3(1+w_i)}.$$

The Friedmann equation is rewritten as

$$H^2 = H_0^2 \left[\Omega_K z^2 + \sum_i \Omega_i (1+z)^{3(1+w_i)}\right],$$

where $\Omega_i \equiv \rho_i / 3M_P^2 H_0^2$ and $\Omega_K = 1 - \sum_i \Omega_i$. Using this equation, find the expression for the luminosity distance $d_L = a_0(1+z) f_K(z)$ as a function of the redshift z.

(4) For simplicity, we consider the flat universe ($K = 0$), filled with Matter and Cosmological Constant. Note that $\Omega_M + \Omega_\Lambda = 1$ in this case. Develop the Mathematica code which does the integration and using it, draw $d_L(z)$ as a function of z for the cases $\Omega_\Lambda = 0,\ 0.3,\ 1$, respectively.

(5) The type Ia supernovae are so bright that they can be observed at very high redshifts. They have roughly a common luminosity independent of the redshift which is well calibrated by their light curves. Hence they are very good standard candles, which can be used to measure luminosity distances. Using the data given in Table 6 of Ref. [3], draw the figure like Figure 1 in which the predictions of cosmological models and the observational data are compared. Note that the luminosity distance data are given as distance moduli

$$\mu_0 = m - M = 5 \log\left(\frac{d_L}{\text{Mpc}}\right) + 25,$$

where apparent magnitude m and absolute magnitude M are logarithmic measure of flux and luminosity, respectively.

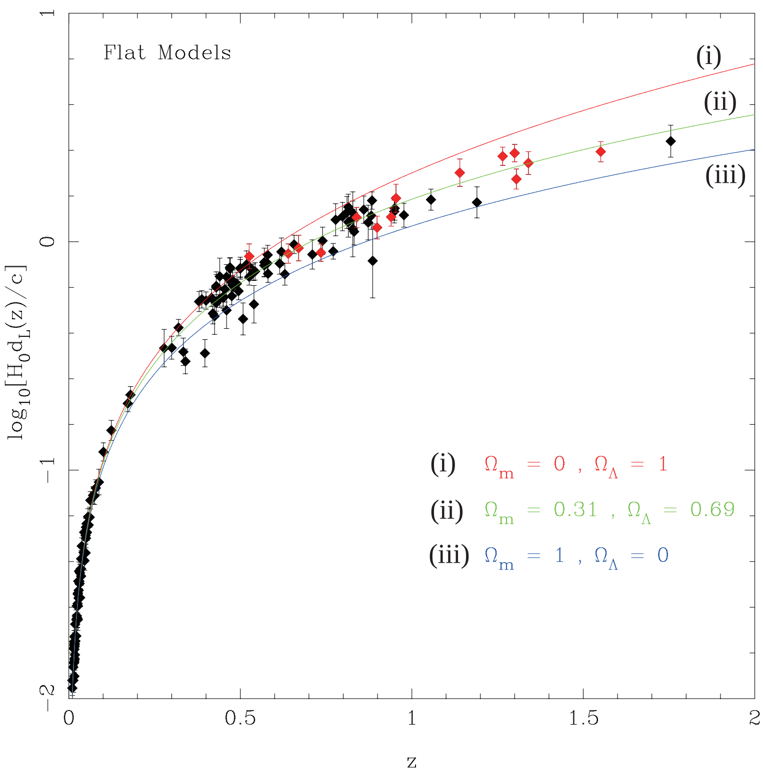

Figure 1: The luminosity distance $H_0 d_L$ versus the redshift z for a flat cosmological model, compared with the observational data. Taken from Ref. [4]

References

[1] A. R. Liddle and D. H. Lyth, "Cosmological Inflation and Large-Scale Structure", Cambridge University Press (2000).

[2] E. J. Copeland, M. Sami and S. Tsujikawa, "Dynamics of dark energy," arXiv:hep-th/0603057.

[3] A. G. Riess *et al.*, "New Hubble Space Telescope Discoveries of Type Ia Supernovae at $z > 1$: Narrowing Constraints on the Early Behavior of Dark Energy," arXiv:astro-ph/0611572.

[4] T. R. Choudhury and T. Padmanabhan, Astron. Astrophys. **429**, 807 (2005) [arXiv:astro-ph/0311622].