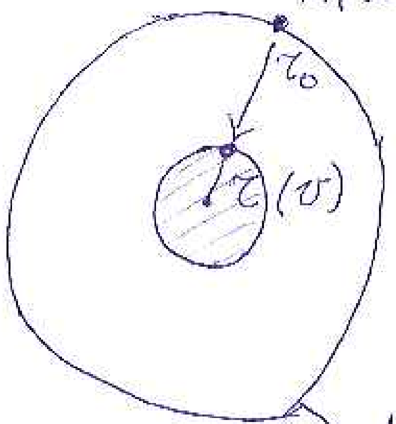
Gravitational collapse: stars to black holes

- collapse of gravitational cloud (no pressure, adiabatic)



collapse from $r_0 \to r$

$$\Delta \text{ kin energy} = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2$$

$$\Delta \text{ pot energy} = \frac{GmM}{r} - \frac{GmM}{r_0}$$

integrate dt from $r_0 \to r$ to determine free fall time t_{FF}

$$\int_{r_0}^{r} dt \equiv t_{FF} = \int_{r_0}^{r} dr \left(\frac{dr}{dt}\right)^{-1} = \int \left(\frac{2GM}{r} - \frac{2GM}{r_0}\right)^{-\frac{1}{2}} dr$$

do the integral by substitution $r = r_0 \sin^2\theta$

$$t_{FF} = \frac{r_0^{3/2}}{\sqrt{2GM}} \int_{\frac{\pi}{2}}^{0} \left[\frac{1}{\sin^2\theta} - 1\right]^{-\frac{1}{2}} \left[2\sin\theta\cos\theta\, d\theta\right]$$

$$t_{FF} = \frac{2r_0^{3/2}}{\sqrt{2GM}} \underbrace{\int_{\pi/2}^{0} \sin^2\theta \, d\theta}_{\pi/4} = \left(\frac{3\pi}{32G\rho}\right)^{1/2} \sim \text{escape velocity}$$

$$\rho = \frac{M}{\frac{4}{3}\pi r_0^3}$$

- when is equilibrium achieved?

$$E_{kin} = E_{Grav}$$

E_{kin} ↑ heated gas; E_{Grav} ↑ for sphere of radius r and mass M

$$\frac{3}{2} N kT \cong G \frac{M^2}{r_{crit}}$$

$r_{crit} = r_{equilibrium}$

N ↑ $\frac{M}{m}$

$$r_{crit} = \frac{2GMm}{3kT} = \frac{3}{2}\left(\frac{kT}{2\pi G\rho m}\right)^{1/2}$$

$$\rho = \frac{M}{\frac{4}{3}\pi r_{crit}^3}$$

substitute M and solve for r_{crit}

$$\rho_{crit} = M / \left(\frac{4\pi}{3} r_{crit}^3\right)$$

$$\rho_{crit} = \frac{3}{3\pi M^2}\left(\frac{3kT}{2mG}\right)^3$$

Example: first structures to form after decoupling.

$m = m_H = 0.94\ GeV$

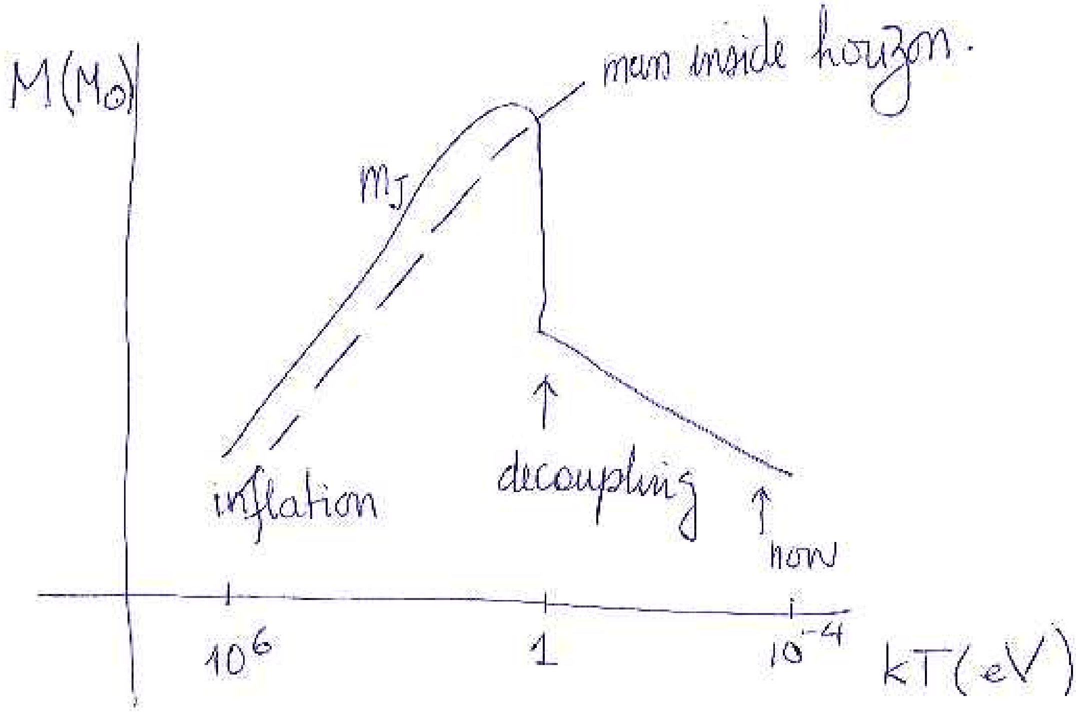
$T \cong$ ~~1eV~~ (13.6 eV → 0.3 eV see cosmology)
$\cong 20K$

$M \simeq 10^4 M_\odot \simeq 10^5$ stars

$r_{cr} = 0.34\ kpc$

Answer: globular clusters.

(stars formed from density fluctuations requiring 10^8 times higher density)



Jeans mass

- smallest mass that can overcome the pressure of radiation and contract under gravity (kinetic energy)

- smaller structures cannot collapse; for less mass, outward pressure wins. Larger can collapse.

→ **dynamics**

- collapse starts with an upward fluctuation of the density in part of it

- density perturbations support sound waves of velocity v_s in the cloud of diameter d

$$\boxed{\lambda_J \cong t_{FF} v_s \simeq r_{crit}} \simeq v_s \left[\frac{\pi}{G\rho}\right]^{1/2}$$

if $d < \lambda_J$ sound waves are standing waves of wavelengths related to d.

$d > \lambda_J$ sound waves do not cover the cloud and perturbations remain that are not erased (brought to equilibrium). cloud can collapse further

examples: CMBR, dark matter with λ_J of order clusters of galaxies

For neutral hydrogen where the Newtonian physics that we used is approximately valid

$$v_s^2 = \frac{\partial P}{\partial \rho} = \gamma \frac{kT}{m}$$

$\gamma \rightarrow \frac{5}{3}$ ratio of specific heats

$$\lambda_J = v_s \left[\frac{\pi}{G\rho}\right]^{1/2} = \left[\frac{5\pi}{3} \frac{kT}{G\rho m}\right]^{1/2} \simeq r_{crit} \quad \nabla_0$$

$\left[\frac{\pi}{G\rho}\right]^{1/2} \rightarrow t_{FF}$

Stars

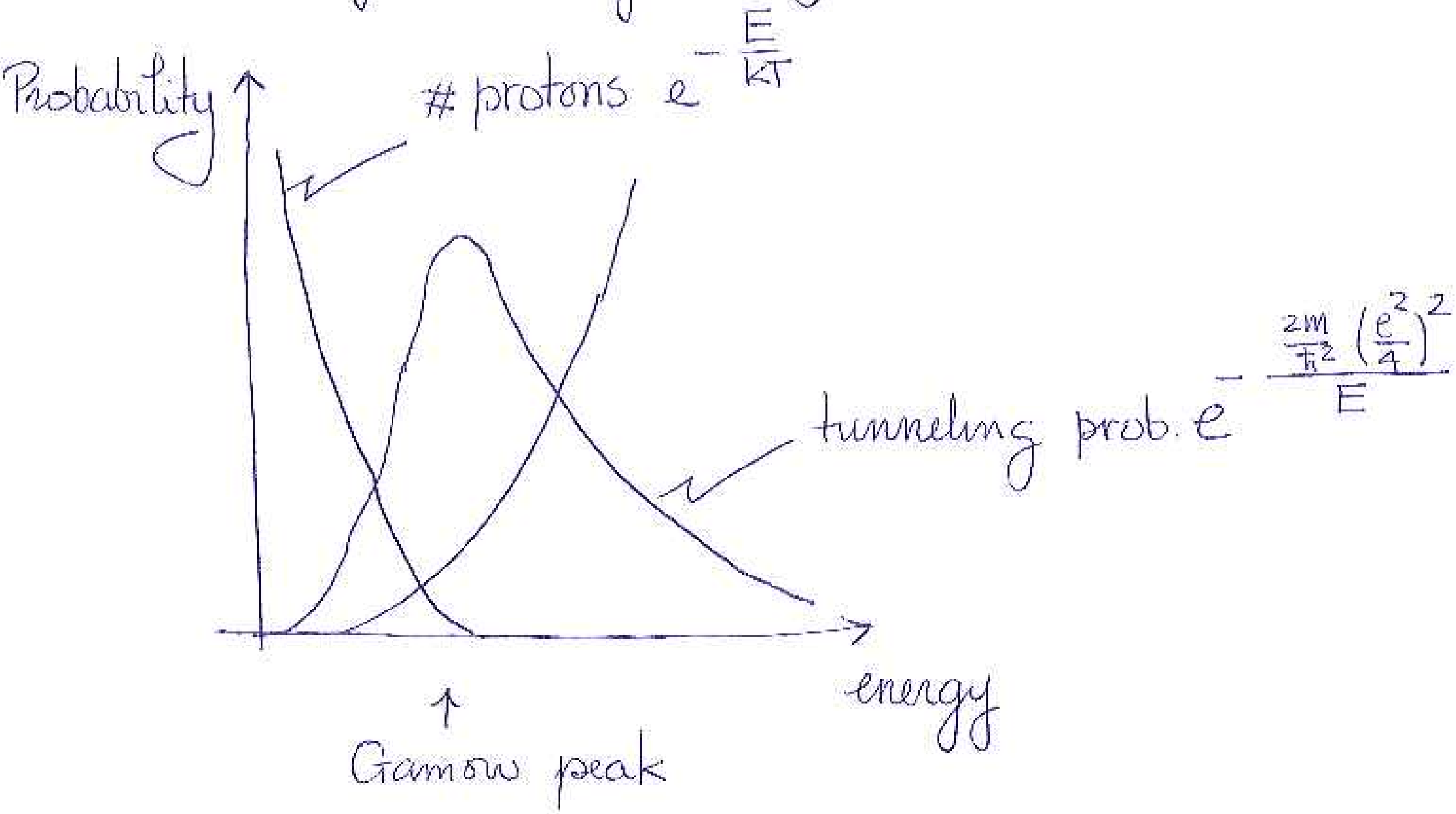
- condense out of clouds of gas to "Jeans" equilibrium (protostar)
- equilibrium between pressure and gravitational energy
- further collapse when hydrogen is ionized by collisions
- nuclear burning starts

$$p + p \to d + e^+ + \nu_e + 0.32\ \text{MeV}$$

Coulomb barrier $\quad V = \frac{e^2}{4\pi}\frac{1}{r} \qquad r \simeq 1\ \text{fm}$

$$V \simeq 0.6\ \text{MeV} \gg kT \sim 1\ \text{keV}$$

protons fuse by tunelling through V



σ fusion/collision $\simeq 10^{-20}$ because of the weak interaction
$\rightsquigarrow$ $\odot$ can burn billions of years

- sun burns mostly by

$$4p \rightarrow {}^{4}He + 2e^{+} + 2\nu_e + 26.73\,MeV$$

1.6% another cycle involving C, N, O (requires higher core temperature)

- hydrogen burning stops, core contracts and heats up, helium burning starts

difficult because there are no stable $A = 5, 6, 8$ elements

$${}^{4}He + {}^{4}He + {}^{4}He \longrightarrow {}^{12}C^{*} \rightarrow {}^{12}C + 2\gamma$$

$${}^{4}He + {}^{12}C \longrightarrow {}^{16}O + \gamma$$

(Hoyle)

- successively the star (if massive enough) will burn

$C \rightarrow Ne \rightarrow O \rightarrow Si \rightarrow Mg \rightarrow Fe$ (most stable nucleus!)

- burning stops: gravitational energy → kinetic energy

- supernova