Generalized Geometrical Objects in Metric Spaces

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Abstract

I investigate some questions resulting from a simple definition of a straight line in an arbitrary metric space. The triangle inequality is used as a "line equality," and defines the line. I try to reproduce some of the definitions for simple geometrical objects and look at the differences from the Euclidian. This definition of a line segment is similar to but not the same as a geodesic.

1 Introduction

A traditional metric space is defined by a set and a mapping ρ from pairs of elements in that set to the reals such that:

1. $\rho(x, y) \geq 0$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) = 0 \iff x \equiv y$
4. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

It is desirable, since spacetime is not a traditional metric space, to extend this study into modified metric spaces which do not satisfy the first criterion. First things first, though–traditional metric spaces are easier to understand.

Some of the properties the metric space can have are those of being open or closed or neither, of being bounded, of being compact, and so on. Different metrics defined over the same space may result in metric spaces with different properties.

1.1 Examples of Metric Spaces

I believe it is important to have some examples available to hand on which one may test various hypotheses and get a feeling for the territory. For simplicity in most of these I impose various metrics on R^2.

1. The simplest metric space is the trivial metric, in which $\rho(x, x) = 0$ and $\rho(x, y) = 1$ when $x \not\equiv y$.
2. Two more simple spaces are R^2 with the metric $\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$: (the 'as the crow flies' metric), and
3. R^2 with the metric $\rho(x, y) =| (x_1 - y_1) | + | (x_2 - y_2) |$: (the 'city streets' (or taxi-cab or L_1) metric).
4. R^3 with a hybrid metric $\rho(x, y) = \sqrt{(x_3 - y_3)^2 + (| (x_1 - y_1) | + | (x_2 - y_2) |)^2)}$ is interesting. Call it a 'city crow' metric–like a bird constrained to fly between skyscrapers, but able to move vertically without hindrance.
5. Another is the distance between two points on the unit sphere. If two points have polar coordinates θ_1, ϕ_1 and θ_2, ϕ_2 then in the 'great circle' metric the distance between them is given by $\sin(\theta_1)\sin(\theta_2)\cos(\phi_1 - \phi_2) - \cos(\theta_1)\cos(\theta_2)$.
6. Consider R^2 with the point $(0, 0)$ missing. Let it have coordinates (r, θ), and define the metric such that for points x_1 and x_2, $\rho(x_1, x_2) = \min(\int_{P_{x_1 x_2}} ds/r)$, where the integral is taken over some path $P_{x_1 x_2}$ from x_1 to x_2. The result (call it the '2D potential

well metric') has several cases. If $x_1 = (r_1, \theta)$ and $x_2 = (r_2, \theta)$ (the same θ), then $\rho(x_1, x_2) = |\ln(r_1/r_2)|$. If the radii are the same, then $\rho(x_1, x_2) = |\theta_2 - \theta_1|$. Otherwise

$$\rho(x_1, x_2) = \sqrt{((\theta_2 - \theta_1)^2 + (\ln(r_2/r_1)^2)}$$

7. Similarly, consider R^3 with $(0, 0, 0)$ missing and define $\rho(x_1, x_2) = \min(\int_{P_{x_1 x_2}} ds/r)$, where the integral is taken over some path from x_1 to x_2. Call this the '3D potential well metric.'

8. Consider the familiar space of joined planes consisting of two copies of R^2, an 'upper' and a 'lower', which are joined along the negative x-axis such that the upper -y joins the lower +y, and the lower -y joins the upper +y. This space may be parameterized by r and θ where θ runs from 0 to 4π. The distance I select is the obvious extension of the 'crow flies' distance: For two points (r_1, θ_1) and (r_2, θ_2), if $|\theta_1 - \theta_2| < \pi$ then the distance is familiar: $\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}$, but if $|\theta_1 - \theta_2| > \pi$ then the shortest distance between the two points is to $(0, 0)$ and back, and so must be $r_1 + r_2$.

9. Consider the metric space (call it Dis4) over 4 points $\{A, B, C, D\}$, with distances between them defined by:

	A	B	C	D
A	0	2/5	1/3	1
B	2/5	0	1/4	3/5
C	1/3	1/4	0	2/3
D	1	3/5	2/3	0

(1)

10. For a nice pathological case, consider the real interval $[0, 2]$, with $\rho(a, b) =| a - b |$ if a and b are rational, and $= 1$ if either of a or b is not rational. Call this Rat2.

11. Those who've walked in swamps know that there are solid bits to walk on, but getting yourself unstuck to get there is hard. In remembrance of swamp-walking, consider a new metric for the real line $\rho(x, y) = f(x, y)\|x - y\|$ where

$$f(x, y) = \begin{cases} 1 & x \notin [0, 2] \cup y \notin [0, 2] \\ 1 + \epsilon^2(x^2 - 2x)(y^2 - 2y) & x \in [0, 2] \cap y \in [0, 2] \end{cases} \tag{2}$$

12. Extend the taxicab metric to 3 dimensions and call it the Office metric.

13. Given a space and a metric ρ on it, define a derived metric ρ' such that $\rho'(a, b) = \rho(a, b)/(1 + \rho(a, b))$. Its values lie in $[0, 1)$.

2 Line Segments and Lines: Definition

In a familiar line segment, the distance from an end-point to a point in the middle plus the distance from that point to the other end-point is equal to the distance between the end-points. This seems to extend very naturally to traditional metric spaces, namely:

$$S_{ab} \equiv \{x \mid \rho(a, x) + \rho(x, b) = \rho(a, b)\} \tag{3}$$

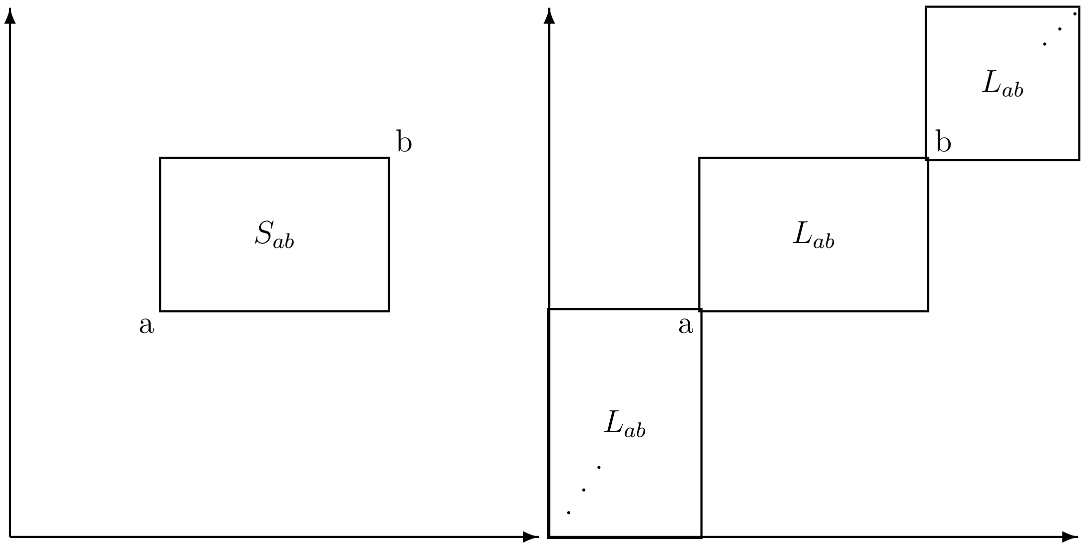

Figure 1: S_{ab} and L_{ab} in taxicab metric

It is simple to extend this definition to an entire line defined by two points:

$$L_{ab} \equiv \{x \mid \rho(a,x) + \rho(x,b) + \rho(a,b) = 2 * \max(\rho(a,b), \rho(a,x), \rho(x,b))\} \tag{4}$$

2.1 Specific Examples of S_{ab}

Obviously if $a \equiv b$ then $S_{aa} = \{a\}$ and L_{aa} is the entire set. It is also immediate that the trivial metric does not have any interesting line segments ($S_{ab} = \{a, b\} = L_{ab}$).

The 'as the crow flies' metric behaves just as expected, but in the 'taxicab' metric a line segment is, in general, a rectangle with opposite corners at the points a and b, and thus has what one might term 'width.' While it is tempting to think of width as the interior of the line segment (and indeed the 'taxicab' metric line segments usually do have an interior), it isn't always possible to define an "interior", as may be seen from the 'city crow' metric.

A line in the 'taxicab' metric consists of the line segment (usually looks like a rectangle) plus the quadrants tangent to the points a and b and to S_{ab}, as shown in Figure 1.

The 'great circle' metric has line segments which are great circle arcs joining the points a and b, unless these points are exactly opposite each other on the sphere, in which case the entire surface of the sphere is the line segment. A line consists of this arc plus an arc from b to a point exactly opposite from a on the sphere, and from a to a point exactly opposite to b on the sphere.

In the '2D potential well' metric space, if the two points have the same θ coordinate, the line segment is a section of radius connecting them. If they have the same radius, the segment is the arc connecting them of the circle they lie on–unless they are exactly opposite in θ, in which case the entire circle is the line segment. If they are neither, then there is a simple equation linking the radius and θ of the arc connecting them: $\theta = ((\theta_2 - \theta_1)/\ln(r_2/r_1))\ln(r/r_1)$. Note again that if the θ's are exactly opposite to each other, there are two arcs, one to each side of the point $(0, 0)$, which are mirror images of each other.

In general, if the metric is defined so that the 'distance' between two points a and b is the minimum of the weighted integral over the path between them (a minimum path), then that path is part of the S_{ab}, by construction.

In the 'swamp' metric if the points a and b are less than 0 greater than 2, then S_{ab} consists of the same set of points you would have using the usual metric on the real line: $[a, b]$. However, if either point is within the interval $(0, 2)$ then $S_{ab} = \{a, b\}$; only the two endpoints are in the segment.

In the "derived" metric we have no non-trivial line segments. If $x \equiv \rho(a, c)$ and $y \equiv \rho(c, b)$ and $z \equiv \rho(a, b)$, then $x + y \geq z$, and we have

$$1 + \frac{1}{1+z} = \frac{1}{1+x} + \frac{1}{1+y} = \frac{2+x+y}{1+x+y+xy} = 1 + \frac{1-xy}{1+x+y+xy} \tag{5}$$

from which we find

$$1 + z - xy - xyz = 1 + x + y + xy \geq 1 + z \tag{6}$$

so x or y must be 0 and only the trivial line segments are possible.

2.2 S_{ab} is Closed

One obvious question is: Is S_{ab} closed, open, or neither? The familiar line segment on the 'crow flies' metric is, of course, closed.

If S_{ab} is a finite set, then it trivially contains its limit points and is closed. Suppose it is not. Select a limit point of S_{ab} designated as z and assert that it is not in S_{ab}. By the definition of a limit point in a metric space, for any $\epsilon > 0$ there exists within S_{ab} some point y such that $\rho(z, y) < \epsilon$.

Because z is not in the segment, $\rho(a, z)+\rho(z, b) > \rho(a, b)$. Let $\delta \equiv \rho(a, z)+\rho(z, b)-\rho(a, b)$. Select $\epsilon < \delta/2$, and as noted above we can find q within S_{ab} such that $\rho(z, q) < \epsilon$. We know that

$$\rho(a, z) \leq \rho(a, q) + \rho(z, q) \leq \rho(a, q) + \epsilon$$

and

$$\rho(z, b) \leq \rho(q, b) + \rho(z, q) \leq \rho(q, b) + \epsilon$$

We can combine these to give

$$\rho(a, b) + 2\epsilon = \rho(a, z) + \rho(z, b) < \rho(a, q) + \epsilon + \rho(q, b) + \epsilon < \rho(a, b) + 2\epsilon$$

which is a contradiction. Therefore S_{ab} must be closed.

2.3 S_{ab} is bounded

This is easily seen. Let $w \equiv \rho(a, b)$; then defining the ball to be

$$B(q, r) \equiv \{x \mid \rho(q, x) \leq r\} \tag{7}$$

clearly each $x \in S_{ab}$ satisfies $x \in B(a, w)$, and thus S_{ab} is bounded. We needn't restrict ourselves to the endpoints, of course. For each $x, y \in S_{ab}$, we have

$$\rho(x, y) \leq \rho(x, b) + \rho(y, b)$$

$$\rho(x, y) \leq \rho(x, a) + \rho(y, a)$$
$$\implies \rho(x, y) \leq \rho(a, b)$$

and the distance between any two points within S_{ab} is less than or equal to $\rho(a, b)$, and $S_{ab} \subset B(x, w)$.

2.4 S_{ab} Contains its Sub-segments of the form S_{ax}

Suppose that there is some point q in S_{ab}. Is it true that $S_{aq} \subset S_{ab}$?

Suppose $\exists y \in S_{aq}$. Then $\rho(a, y) + \rho(y, q) = \rho(a, q)$, and since $\rho(a, q) + \rho(q, b) = \rho(a, b)$, then $\rho(a, y) + \rho(y, q) + \rho(q, b) = \rho(a, b)$. Since we have, by definition of a metric space, $\rho(y, q) + \rho(q, b) \geq \rho(y, b)$, we get

$$\rho(a, b) = \rho(a, y) + \rho(y, q) + \rho(q, b) \geq \rho(a, y) + \rho(y, b) \geq \rho(a, b)$$

Since we are bounded above and below by $\rho(a, b)$ the $\geq$ must be $=$, and we see that $\rho(a, y) + \rho(y, b) = \rho(a, b)$, and hence $y \in S_{ab}$.

Thus $S_{aq} \subset S_{ab}$ if $q \in S_{ab}$.

If we know that for a set of points x_i

$$\rho(a, x_1) + \Sigma_i \rho(x_i, x_{i+1}) + \rho(x_{n+1}, b) = \rho(a, b)$$

we can iterate combining the distances to likewise show that all of the x_i are in S_{ab}.

It is not true in general that if $x, y \in S_{ab}$ that $S_{xy} \subset S_{ab}$. For a simple counterexample, consider the '2D potential well metric' with points on opposite sides of the singularity, e.g. $(1, 0)$ and $(-2, 0)$. The line segment consists of two branches around the singularity. If you take one x from one branch and y from another the resulting segment between them is almost entirely disjoint from the original.

2.5 Widths and Measures in a Line Segment

Define a mapping from S_{ab} and the real number t to a set of points in the segment which are at distance t from the point a. Clearly t has to lie within $[0, \rho(a, b)]$.

$$G(S_{a,b}, t) \equiv \{q \mid q \in S_{ab} \cap \{v \mid \rho(a, v) = t\}\}$$

To keep the notation simple, I'm assuming that the a which is the reference for this distance is the first point mentioned in $S_{a,b}$, instead of explicitly specifying it. This set of points may be empty for some values of t, or have one point only, or many points. For $t = 0$ we trivially have $G(S_{a,b}, 0) = \{a\}$, and also $G(S_{a,b}, \rho(a, b)) = \{b\}$. Given a line segment S_{ab} and mapping

$G(S_{a,b}, t)$, we can define a real function from $[0, \rho(a, b)]$ to $[0, \rho(a, b)]$ that describes a type of width of the line segment.

$$W(S_{a,b}, t) \equiv \max(\rho(c, d)) \mid c, d \in G(S_{a,b}, t)$$

It is possible that there are no points in the set for a given t_i, in which case define $W(S_{a,b}, t_i) = 0$. If there is only one point the value is also 0, of course. To be able to talk about this easily we need another function that helps precisely define continuity over $[0, \rho(a, b)]$:

$$Cont_{a,b}(t) \equiv \begin{cases} 1, & G(S_{a,b}, t) \neq \emptyset; \\ 0, & G(S_{a,b}, t) = \emptyset \end{cases} \tag{8}$$

Since $Cont_{a,b}(t)$ will always be 1 when $t = 0$ or $t = \rho(a, b)$, if it is discontinuous then there are gaps in the mapping. I call the segment $Cont$ if $Cont_{a,b}(t) = 1$ everywhere.

Another definition of width is possible; call it the thickness. Define the thickness at the distance x from a as the largest sphere C one can have entirely within $S_{a,b}$ centered in the set of points at distance x

$$Tx(S_{ab}|x) \equiv max(r|C(y, r) \subset S_{ab}, y \in H(S_{ab}|x)) \tag{9}$$

The thickness of the segment itself is

$$Tx(S_{ab}) \equiv max(Tx(S_{ab}|x)) \tag{10}$$

A line segment can have a non-zero width but a thickness of 0. For example, consider the Office metric for the case in which the points a and b share the same x coordinate. The resulting S_{ab} is a plane, which in general has a non-zero width, but the thickness is 0.

2.5.1 Geodesics

If S_{ab} is $Cont$ then we can find at least one mapping g from the real set $[0, \rho(a, b)]$ to S_{ab} such that for every r in the interval there is one point in S_{ab}. It need not be true that the distance between points r and $r + \epsilon$ is small: it may be relatively large. Consider the pathological "swamp" metric as an example.

However, if the mapping g does satisfy $\rho(g(r), g(r + \epsilon)) = \epsilon$ then we can call g continuous in the usual sense, and consider it a geodesic in the usual sense of the term. There might be amusing complications if the equality is replaced with an ϵ/δ construction instead.

There can be more than one distinct continuous mapping g_i. These are separated from each other in the following sense. Pick a distance from a at which two are distinct and call it r. Let $g_1(r) = x_1$ and $g_2(r) = x_2$, and $\rho(x_1, x_2) = K > 0$. There exists $\epsilon > 0$ such that for all $q \in (r - \epsilon, r + \epsilon)$, $\rho(g_1(q), g_2(q)) > 0$.

Suppose this were not true, and there were some q in that interval for which $g_1(q) = y_1 = y_2 = g_2(q)$. We have $\rho(x_1, y_1) < \epsilon$ and $\rho(x_2, y_1) < \epsilon$. We therefore must have $2\epsilon > \rho(x_1, y_1) + \rho(x_2, y_1) > \rho(x_1, x_2) = K$, but since ϵ is arbitrarily small and K is finite this is a contradiction. (Also true for the ϵ/δ construction variant I mention above.)

There are always at least two intersections between two such geodesics: the end points. If there are countably many then as usual one can find a sequence with a limit λ in the real interval $[0, \rho(a, b)]$. For points in each geodesic close to the map for λ the maximum distance $K = \rho(x_1, x_2)$ will have to become arbitrarily small.

2.6 Partitionable and Well-ordered

If a line segment has the property that for each point $c \in S_{ab}$, $S_{ab} = S_{ac} \cup S_{cb}$ then that line segment is **partitionable** . This is a fairly strong requirement. Segments consisting of only three points are trivially partitionable .

It is easily seen that $W(S_{a,b}, t)$ must be 0 everywhere for a partitionable segment. If it is non-zero for some t then there exist two points x and y in S_{ab} such that $\rho(a, x) = \rho(a, y) = t$ (and also $\rho(x, b) = \rho(y, b)$). If we could partition S_{ab} then y would have to be in either S_{ax} or S_{xb}: say S_{ax} without loss of generality. This requires $\rho(a, y) + \rho(y, x) = \rho(a, x) = \rho(a, y)$, which would imply $\rho(y, x) = 0$ which contradicts the assumption that there are two distinct points.

A segment which has $W(S_{a,b}, t)$ always 0 call 'well-ordered.' If this is true for all line segments in the space, then call the metric space 'well-ordered' too. It can happen that a line segment will have multiple branches, each of which, considered separately, is well-ordered; but the whole segment is not. Line segments of the 'taxi-cab' are generally not well-ordered, though some are for selected a and b.

'Well-ordered' is an analog of the usual definition of a line segment, which is the set of points $(1 - t)x + ty : 0 \leq t \leq 1$. At first glance partitionable and 'well-ordered' seem to be closely related, and in fact partitionable implies 'well-ordered.'

Given $x, y \in S_{ab}$, if S_{ab} is partitionable we must have $y \in S_{ax}$ or $y \in S_{xb}$. Without loss of generality assume the first case, $\rho(a, y) + \rho(y, x) = \rho(a, x)$. If $x \neq y$ we have $\rho(y, x) > 0$ and thus $\rho(a, y) \neq \rho(a, x)$. Since x and y were selected arbitrarily, this means that no two points have the same distance from a and thus $W(S_{a,b}, t) = 0$. So partitionable implies 'well-ordered.'

On the other hand, well-ordered does not imply partitionable, as shown by a counterexample. The metric Dis4 has a non-trivial line segment in $S_{A,D}$, which consists of the entire set. This segment is clearly well-ordered. It is not partitionable . Thus well-ordered does not imply partitionable .

A segment can be well-ordered and *Cont* but not partitionable . Consider the 'swamp' metric and the segment S_{02}. It consists of all the points in $[0, 2]$, has no gaps, and no two points have the same distance from 0. However, for any x in S_{02} where x is neither 0 nor 2, the union of the segments S_{0x} and S_{x2} consists of only 3 points: $\{0, x, 2\}$.

A weaker condition than partitionable can be useful. Call a line segment 'locally partitionable ' if, for all but a finite number of points, we have

$$q \in S_{ab}, \exists \epsilon > 0 \mid (B(q, \epsilon) \cap S_{aq}) \oplus (B(q, \epsilon) \cap S_{bq}) = B(q, \epsilon) \cap S_{ab}$$

A line segment can bifurcate, but so long as there are only a finite number of these bifurcation points it can be locally partitionable . That $S_{aq} \subset S_{ab}$ if $q \in S_{ab}$ we know from section 2.3. Along with this we can define the property of being 'locally well-ordered' in a natural way.

If $W(S_{ab}) > 0$, some of the $W(S_{ab}, q) > 0$. Either $W(S_{ab}, q) > 0 \forall q \notin \{a, b\}$, or there exists some q where it is $= 0$. In the first case it may be possible to parameterize something more like a conventional line segment, one of which contains a given point $q \in S_{ab}$ while others do not.

2.7 When Does S_{ab} Contain its Sub-segments Generated from Internal Points?

We already saw that S_{ab} does not always contain sub-segments created from points within it, but it is interesting to determine when it does and when it does not.

If for some x, y in S_{ab}, $S_{xy} \not\subset S_{ab}$, then clearly x is not in S_{ay} or S_{by}, and likewise y is not in S_{ax} or S_{bx}, or else by the result above $S_{xy} \subset S_{ay} \subset S_{ab}$ (for example). Thus if $S_{xy} \not\subset S_{ab}$, then $S_{ab} \neq S_{ax} \cup S_{xb}$, since we have a point y which is not in either of the two sub-segments. S_{ab} is not partitionable .

Notice that the converse is not true: S_{ab} not partitionable does not imply that there exists $\{x, y\} \in S_{ab}$ with $S_{xy} \not\subset S_{ab}$. The 'taxi cab' metric space provides a simple counter-example: each line segment contains all segments creatable from points within it, but it is not a simple sum of two sub-segments S_{ax} and S_{xb}.

A line need not contain all line segments generated by the points within it. This is obvious from considering the 'great circle' metric, where a line is (in general) an arc extending more than half-way round the sphere. Thus there are two points $\{x, y\}$ in L_{ab} which are exactly opposite to each other, and S_{xy} is the entire surface of the sphere–which is not contained in L_{ab} in general.

The '2D potential well' metric space offers an example of a space in which not all line segments are partitionable , and in fact in which there exist line segments which do not contain line segments generated from points within themselves. It is easy to see that most of the line segment arcs in this space are partitionable , but when the points are at different radii and are opposite each other in θ, the line segment consists of two non-circular arcs joining the points. Clearly the S_{ac} and S_{cb} formed from using a point in one of these arcs will not generate any points in the other arc, and this S_{ab} is not partitionable . In addition, if you take a point from one of the arcs and a point from the other, the line segment formed between them will, far from being a subset of S_{ab}, only intersect S_{ab} in two points.

At the moment all I have are examples to outline the nature of the questions.

2.8 Nearest Point in a Line Segment

Given a line segment S_{ab} (not equivalent to the entire space) and a point c not in it, then define the nearest points as

$$N(S_{ab}, c) = \{x \in S_{ab} \,|\, \rho(c, x) = d \,|\, d = \min(\rho(c, y)) \; y \in S_{ab}\}$$

Since S_{ab} is closed, there is at least one point in N.

There can be more than one point in N. For example take the 'great circle' metric space. Consider one S_{ab} which is an arc on the 'equator' and c on a 'pole.' In this case $N(S_{ab}, c)$ is the entire S_{ab}. The 'great circle' metric space was not devised to be pathological.

2.9 Non-trivial Intersection of Line Segments

In many cases two line segments will not intersect at all, or only in a single point. I think this is understood well enough to require no comment.

If the segments have $W(S_{a,b}) = 0$ and $W(S_{c,d}) = 0$ (the maximum of $W(S_{a,b}, t)$), and $a, b \notin S_{c,d}$ and $c, d \notin S_{a,b}$

Assume there are points a, b, c, and d such that $c, d \notin S_{ab}$ and $a, b \notin S_{cd}$. This will not always be possible, of course (as when a and b are on opposite poles of a sphere, with the usual metric on a sphere). Call their intersection

$$I_{a,b|c,d} \equiv S_{ab} \cap S_{cd}$$

Often I will be $\emptyset$, and there are metric spaces in which it is always $\emptyset$ (using the trivial metric, for example), but consider for now the instances when it is not empty, and also not the entire space (as can happen in 1-dimensional spaces).

Clearly I is closed.

It may be of interest to examine when I is connected. To do that we'd need to define connectedness and convexity in this space.

Is $I_{a,b|c,d} = S_{rs}$ for some r and s?

Not always: there is a counterexample in the '2D potential well metric.' Points exactly opposite each other in angle but at different radii have 2-branched line segments, which can have two intersection points with the segment joining a pair of points at the same radius. These don't form even a trivial line segment.

So instead of $I_{a,b|c,d}$, consider the connected subsets of it. Label these (if there are countably many) with i and call them $J^i_{a,b|c,d}$.

Are these $J^i_{a,b|c,d} = S_{rs}$ for some r, s?

Not always. If the intersection part is a single point, that is a trivial line segment and the conjecture is true. However, take as a counterexample the case of the '3D potential well metric' when $\{a, b\} = \{(N, 0, 0), (-N, 0, 0)\}$ and $\{c, d\} = \{(N, \epsilon, 0), (-N, -\epsilon, 0)\}$ where N is large.

The S in these cases take the form of long conical sheaths that reach from each endpoint to meet around the singularity. The intersection of the "conical" portions near one of the endpoint will consist of a couple of loops. If two points are far from the singularity the line segment between them will look much like a line segment in R^3–they cannot generate a loop. Therefore the J cannot be a line segment in this example, and therefore not in general.

Assume for the moment that the union of the two line segments is not the same as the entire space. Given a point q in the intersection $J^i_{a,b|c,d}$, can we create a non-empty ball

$B(q, R) \equiv \{x \mid \rho(x, q) \leq R\}$ which contains points not in the intersection of the two line seqments? (In other words, extending beyond the intersection blob.) Assume we can, for R greater than some R_{min} (though the cases in which one cannot might have interesting pathologies). Now consider $U \equiv B(q, R) \cap S_{ab}$. Into how many continuous convex parts is it divided?

If there are 0, 1 or more than 2 convex parts I'm not ready to deal with the situation right now. If there are 2, then let's proceed.

$U \equiv U_1 \cup U_2$ where U_1 and U_2 are the two convex parts. Let $p_1 \in U_1 \cap \overline{J^i_{a,b|c,d}}$ and $p_2 \in U_2 \cap \overline{J^i_{a,b|c,d}}$. If either of these two sets is empty, we again have a curious situation which I'll ignore for the moment. Now find a point Q (if it exists) in S_{cd} such that $Q \notin S_{ab}$ and $\rho(Q, i) > R$. Let $W_1 = \min(\rho(p_1, Q)), p_1 \in U_1 \cap \overline{J^i_{a,b|c,d}}$, and W_2 be the corresponding minimum for p_2. Use these to define $f(Q, R) \equiv W_1^2/(W_1^2 + W_2^2)$. Now define $f_L(R) = \min(f(Q, R))$ and $f_H(R) = \max(f(Q, R))$. If these converge such that $\lim_{R \to R_{min}} f_H(R) - f_L(R) = 0$, then we can define a unique *angle* of intersection, which is given by $\sin(\theta/2) = \lim_{R \to R_{min}} f_H(R)$. It may be zero; perhaps in some metrics even always zero. For the standard Euclidean metric the definition returns the usual value for θ.

2.10 Dimensions

Dimensions aren't always easy to define, but I at least need to have something that allows me to exclude trivial cases. There are two obvious definitions of a 1-dimensional space: There exist two points a and b for which L_{ab} is the entire space; or alternatively, for all distinct points a and b, L_{ab} is the entire space. *It isn't clear which is most useful yet, and I have not taken up such fine points as "except for a finite number of points" or "except for a finite set of disconnected regions."*

An alternative approach is iterative. Consider some local region of the metric space. If for every point x in S_{ab} (excluding a and b) there exists a ball $B(x, \epsilon) \subset S_{ab}$, the space is locally 1D.

In section 3 I discuss ways of defining "inside" for a generalized triangle.

If I find a point $c \notin S_{ab}$ and define a triangle ${T^1}_{a,b,c}$ and if for every point $x \in {T^1}_{a,b,c}$ (not in any of the S_{ac}, etc) there exists a ball $B(x, \epsilon) \subset {T^1}_{a,b,c}$, the space is locally 2D.

If I find a point $d \notin {T^1}_{a,b,c}$ and define a 'tetrahedron' ${T4^1}_{a,b,c,d}$ in the analogous way to T^1, and if for every point $x \in {T4^1}_{a,b,c,d}$ (not in any of the ${T^1}_{a,b,d}$, etc) there exists a ball $B(x, \epsilon) \subset {T4^1}_{a,b,c,d}$, the space is locally 3D.

2.11 Hyperbolic

A Gromov product in a metric space is defined by

$$(a, b)_c = \frac{1}{2}(\rho(a, c) + \rho(c, b) - \rho(a, b)) \tag{11}$$

This will always be greater than or equal to zero. If $c \in S_{a,b}$, then $(a, b)_c = 0$.

A Gromov-hyperbolic metric space satisfies, for all a, b, c, and d in the space,

$$(a, b)_c \geq \min((a, d)_c, (d, b)_c) - \delta \tag{12}$$

If all segments in the metric space are trivial, then the space will be Gromov-hyperbolic.

3 Planes

There are four obvious definitions of a 'plane' defined by 3 points. In the usual Euclidean space these are equivalent, but not in general.

Require that the three points a, b, and c are not in the same line. We can try to use a line defined by two of the points and a third point not on the line, as in

$$P^s_{L_{ab},c} \equiv \{y \mid y \in L_{cr};\, r \in L_{ab}\} \tag{13}$$

Sometimes one will have $P^s_{L_{ab},c} \equiv P^s_{L_{ac},b} \equiv P^s_{L_{bc},a}$, but this need not be true in general. A more symmetric definition is better:

$$P^2_{a,b,c} \equiv \{y \mid y \in L_{Xr};\, X \in \{a, b, c\},\, r \in L_{ab} \cup L_{bc} \cup L_{ac}\} \tag{14}$$

In Euclidean geometry one can get away with an even smaller definition; though this can produce amusing unexpected gaps in the coverage in more general cases:

$$P^1_{a,b,c} \equiv \{y \mid y \in L_{Xr};\, X \in \{a, b, c\},\, r \in S_{ab} \cup S_{bc} \cup S_{ac}\} \tag{15}$$

Alternatively we can use a union of all lines generated from all points in the lines generated by the three points, as in

$$P^4_{a,b,c} \equiv \{y \mid y \in L_{sr}; s, r \in L_{ab} \cup L_{bc} \cup L_{ac}\} \tag{16}$$

Once again, in the Euclidean metric we can get away with a smaller definition involving lines between points on the line segments generated by the three points.

$$P^3_{a,b,c} \equiv \{y \mid y \in L_{sr}; s, r \in S_{ab} \cup S_{bc} \cup S_{ac}\} \tag{17}$$

We can also define a plane-like object by selecting one of the definitions of a 'plane' and generating all lines formed from points within that object; continuing the iteration until we get convergence (if that ever happens!).

An obvious first question is: 'Do these result in the same sets?' The answer is no. A simple counter-example is the 'great circle' metric. Given 3 points, the plane defined by P^1 consists, in general, of the area contained within 6 arcs defined by the points and lines between them–it is NOT the entire sphere, in general. However, P^4, consisting of all lines joining points in any of the lines, must consist of the entire sphere, since any line must include at least two points opposite each other on the sphere, and any line joining two points opposite each other comprises the entire sphere. Thus these are not equivalent definitions.

We have (by construction) that $P^s \subset P^2$ for any order of (a, b, c); and it is obvious that $P^1 \subset P^2$ and $P^3 \subset P^4$. It is not hard to see that $P^1 \subset P^3$ and $P^2 \subset P^4$, and if we iterate P^1 or P^3 as described above, that $P^2 \subset P^1_{iter}$ and $P^4 \subset P^3_{iter}$.

A second question is: 'Does a line partition a plane defined by that line and another point?' This depends on how one defines partition, apparently. In the case of the 'great circle' metric two points on 'opposite' sides of the line in a plane (P^1) defined by another point cannot be joined by a great circle arc which does not intersect the line, but can be joined by a series of arcs which don't intersect it. Of course, P^1 is a deliberately minimal definition.

This needs more work.

4 Inside/Outside

Consider a 'triangle' defined by 3 points, none of which is in a line defined by the other two.

$$T^1{}_{a,b,c} \equiv \{\cup S_{ij} \mid i, j \in \{S_{ab} \cup S_{bc} \cup S_{ac}\}\} \tag{18}$$

Sometimes sweeping out the "angles" from the vertices will be equivalent, but not always:

$$T^2{}_{a,b,c} \equiv \{\cup S_{ij} \mid i \in \{a, b, c\}, j \in \{S_{ab} \cup S_{bc} \cup S_{ac}\}\} \tag{19}$$

The T^2 definition clearly does not give the same set as the T^1 definition when applied to the '2D potential well metric' when the 3 points are spaced around the singularity. The "hole" in the middle is shaped differently between the two. It is possible to take three points from the interior of the original triangle and create a triangle from these with a new and "smaller" "hole" in the middle. One could define a limiting "interior" as the union of all such sub-triangles and their sub-triangles in turn.

What constitutes the inside and what the outside of the triangle? There may in fact be nothing 'inside' in any reasonable sense–for example consider the 'city streets' metric, in which the line segments between the three points completely fill the rectangle defined by the most extreme points with nothing left over to call "inside." I use the definition

$$In_{a,b,c} \equiv \{\cup S_{ij} \mid i, j \in \{S_{ab} \cup S_{bc} \cup S_{ac}\}\} - \{S_{ab} \cup S_{bc} \cup S_{ac}\}$$

for the inside of this 'triangle', with the understanding that it may be empty.

I'd like to know if this definition of a triangle results in a triangle defined by three points which is a subset of the plane defined by those three points. It is consistent with P^3 and P^4, but it isn't clear yet if it works with the other two.

5 Interpretations

For a metric space defined by minimization of some quantity over a path, S_{ab} represents the set of points on 2-stride paths between a and b. Since $S_{a,b}$ defined this way is minimizing over a path of an infinite number of strides, you would expect that it would contain all sub-segments generated by pairs of points within it.

6 Miscellaneous Questions

Given two segments S_{a,b_1} and Sa, b_2, and given two points c_1 and c_2 on each respectively, both the same distance from a, what can we say about the relative sizes of $\rho(c_1, c_2)$ and $\rho(b_1, b_2)$? In the ordinary Euclidean metric the former will be smaller than the latter, but in general you don't know that. As an example, in the '2D potential well metric', if a is on the near side of the peak and b_1 and b_2 are almost on the opposite side; just a little to the left and right respectively, the c points along the arcs may be farther apart or nearer than the end b points, depending on where along the arc you select them.

Given a and b, under what conditions can one find x and y such that S_{xy} is L_{ab}? This is certainly sometimes possible when the space is bounded and closed, as can be seen by considering the unit disk in R^2 with the standard 'crow flies' metric: just pick x, y to lie on the boundary and you have $\{x, y\} = \{a, b\}$, $S_{xy} = L_{ab}$. When the space is unbounded, it is at least sometimes impossible for L_{ab} to be S_{xy}.

Under the different definitions of the triangle, under what conditions can one "triangulate" the metric space? Is a well-defined dimension of the space a prerequisite? Can one readily expand the definition to the equivalents of higher dimensions?

One can easily pathologize a bounded metric space to have a point which cannot lie in the interior of any line segment (ie. not one of the endpoints), by defining the distance between that point and any other to be greater than twice the largest distance between any other two points. However, if the metric is continuous according to some measure (e.g. all S_{ab} are $Cont$), what does this excluded subset look like? Some kind of boundary? Does it have a thickness?

Is there some simple way to determine the boundary of a simplex in the metric space?

7 Families of Metric Spaces That Preserve the Sets Comprising Line Segments

Given a set of points and a metric on these, we consider the set of all line segments. There is a family of metric spaces which result in the same set of line segments. To be precise, using the same a, b results in the same set of points S_{ab} for any of the metrics in this family and for all a and b in the set (or local region of the set). For example if a metric ρ on a set of points results in a set of line segments S, $\rho \to c\rho$ for $c > 0$ results in the same set S.

This can be more extensive than simple scaling. For example, consider R^2 with the metric function $\rho = ((\delta x)^\alpha + (\delta y)^\alpha)^{1/\alpha}$. If $\alpha > 1$, line segments are the same no matter what α is selected, even though circles are quite different. If you use a linear mapping of the x and y onto $x^{'}$ and $y^{'}$ and plug that into the metric function, line segments are unchanged.

In the trivial case of a space consisting of $\{a, b, c\}$ you can get the same set of line segments $\{S_{ab}\}$ with arbitrarily different metrics: e.g. $\rho_1(a, b) = 2, \rho_1(a, c) = 1, \rho_1(c, b) = 1$ and $\rho_2(a, b) = 2, \rho_2(a, c) = 1/4, \rho_2(c, b) = 7/4$.

8 Angle at an Intersection

If we take two lines L_{ab} and L_{cd} that intersect in a non-empty $I_{ab,cd}$, the obvious approach is to try to find the distance between points on each line equidistant from their intersection, and see if the ratio of the distance between those points (Ch) and their distance from the intersection (R) becomes constant with decreasing distance. That gives us a way to define a local angle. Since this is a metric space we have $2R \geq Ch$.

To that end, we need to look at intersection cases. For simplicity of notation, call L_{ab} L_1 and call L_{cd} L_2, and call the intersection I; just assume the other points have been defined already.

I-A	I is a single point
I-B	I is a finite set of points
I-C	I is an infinite set of points
I-D	I contains a disk of radius $\epsilon > 0$

For the moment, only consider case I-A, or a point from I-B in which the point is isolated from the rest by some distance κ. Call the intersection point used here p.

In order to define our limit, the lines L_1 and L_2 should be 'continuous' in some sense. Name the intersection of a disk $D_\epsilon(p)$ with L as $Q_\epsilon(p, L)$.

C-0 continuous: For all $\epsilon > 0$ if $Q_\epsilon(p, L) - \{p\}$ is non-empty.

C-1 continuous: If it is continuous in the C-0 sense and if for all $0 < \epsilon_2 < \epsilon_1 < \kappa$, for some appropriate κ, $Q_{\epsilon_2}(p, L)$ is a proper subset of $Q_{\epsilon_1}(p, L)$.

Assume both lines L_1 and L_2 are C-1 continuous.

One can demand other forms of continuity, such as that for every $\epsilon > 0$ and p on the line, there are points x on the line satisfying $\rho(p, x) = \epsilon$ (C-2), and so on.

Consider now circles $C(p, \epsilon) = \{x|\rho(p, x) = \epsilon\}$ that intersect the lines.

It is possible to have a circle that intersects one but not the other locally continuous line, for some radius less than κ, while at nearby radii both are intersected, by for example removing a point from the space. It would not contain its limit points.

Denote the intersection of a circle and line as $W_\epsilon(p, L) \equiv C(p, \epsilon) \cap L$ In ordinary Euclidean geometry this will have two points in it (recall that p is on the line L).

Pick pairs of points (h_1, h_2) from $W_\epsilon(p, L_1)$ and $W_\epsilon(p, L_2)$ respectively, and from these select a set of pairs for which the distance $\rho(h_1, h_2)$ is minimum. There may easily be more than 1 of these (as in Euclidean geometry). Pick one of these pairs: $(h_{\epsilon,L_1}, h_{\epsilon,L_2})$.

Now find another $\epsilon_2 < \epsilon_1$, and find the same type of pairs as before with the new ϵ. From that set, pick one for which both $\rho(h_{\epsilon_2,L_1}, h_{\epsilon_1,L_1})$ and $\rho(h_{\epsilon_2,L_2}, h_{\epsilon_1,L_2})$ are minimum: assuming this is possible. If a metric is pathological enough C-1 continuity isn't enough to guarantee this.

We are now able to compare the change of distances with radius.

The ratio of $Ch_1 = \rho(h_{\epsilon_1,L_1}, h_{\epsilon_1,L_2})$ to ϵ_1 and the ratio of $Ch_2 = \rho(h_{\epsilon_2,1}, h_{\epsilon_2,2})$ to ϵ_2 should be at least consistent as ϵ_1 decreases. If the ratio converges to some constant value

r as ϵ_1 decreases, we are able to define an angle between the lines in a consistent way. $\theta \equiv 2\arcsin(r/2)$ Since, as noted above, $Ch \leq 2R$, if the limit of $r = Ch_1/\epsilon_1$ exists and is well defined, then the angle θ is also well defined.

8.1 Loosening

How many of the assumptions above can be loosened? If the ratio mentioned converges, with a set of excursions 'of measure 0', that should suffice for a nearly correct angle.

As an example of a pathological metric that offers C-1 continuity for the given lines, consider a Euclidean x-y plane with exceptional points. Distances between all points are the same as before, except for those for which $y = 1/2^n$ for $n > 1$, where the distance is 5 times the usual distance. Lines defined by $L_{(1,1),(-1,-1)}$ and $L_{(1,-1),(-1,1)}$ intersect in $(0, 0)$, and are C-1 continuous with respect to that point. However some points will be missing from both lines, namely those for which $y = 1/2^n$. One could define open bands as the pathological parts instead, in which case the lines L would include their limit points, so merely adding that condition doesn't guarantee that a circle at the intersection of the lines will always intersect both lines.

9 Looking at Balls

Recall that a Ball is $B(a, r) \equiv \{x|\rho(a, x) \leq r\}$, a Sphere is $Sp(a, r) \equiv \{x|\rho(a, x) < r\}$, and a Circle is $C(a, r) \equiv \{x|\rho(a, x) = r\}$.

Define

$$Q(a, r) \equiv \{\cup S_{a,x} \forall x \in C(a, r)\} \tag{20}$$

The metric around this Ball is "well-behaved" if

$$Q(a, r) = B(a, r) \tag{21}$$

If it is not "well-behaved", then $\exists y \in B(a, r)|r - \rho(a, y) < \rho(a, z)\forall z \in C(a, r)$

It isn't hard to kludge up a metric space for which this can be true. For example, pick the ordinary R^2 and pick a point q for which $\rho(a, q)$ is the usual one, but $\rho(x, q)$ is some large number for any $x \neq a$. You can probably also come up with situations in which the boundary Circle has significantly fewer points than Circles of smaller radius.

What requirements will make the metric "well-behaved" almost everywhere?

In a possibly related question, can one have a Sphere around a remote point in which every point but the center is farther from the starting point than the center?

$$R(b, r_1, r_2) \equiv Sp(b, r_2) - B(b, r_1), r_2 > r_1 > 0 \tag{22}$$

Suppose that for every $z \in R(b, r_1, r_2)$ we have $\rho(a, b) < \rho(a, z)$ for some a.

Naturally $\rho(a,z) \leq \rho(a,b) + \rho(b,z)$. Since $\rho(a,z)$ is larger than $\rho(a,b)$ by assumption, $\rho(a,z) = \rho(a,b) + \delta$ with $\delta > 0$. We can then write

$$\rho(a,b) < \rho(a,z) = \rho(a,b) + \delta \leq \rho(a,b) + \rho(b,z) < \rho(a,b) + r_2 \tag{23}$$

Giving

$$\rho(a,z) < \rho(a,b) + r_2 \tag{24}$$

We can't automatically require r_1 or r_2 to be arbitrarily small, but at least this brackets δ, which has to satisfy $\delta < r_2$.

If there is a non-empty line $S_{a,b}$, assume we can find adjust r_2 to find a z in the ring and a y outside the ring but in $S_{a,b}$ such that the distance between z and y is less than ϵ. If so, then because $\rho(a,y) + \rho(y,z) \geq \rho(a,z)$ then $\rho(a,y) + \epsilon > \rho(a,z)$, with ϵ becoming arbitrarily small. But since this brings $\rho(a,z)$ arbitrarily close to a value less than $\rho(a,b)$, we have a contradiction.

Therefore either $S_{a,b}$ is empty in this situation, or we can't pick y and z arbitrarily close together. We know that for y in $S_{a,b}$ (which means $\rho(b,y) = \rho(a,b) - \rho(a,y)$), $\rho(y,z) > \rho(y,b)$.

There is thus some kind of break in $S_{a,b}$, such that for some arbitrarily small ϵ we can find a $y \in Sa,b$ such that $\rho(a,y) + \epsilon < \rho(a,b)$, but for which there are no $x \in S_{a,b}$ such that $\epsilon < \rho(a,x) - \rho(a,y) < 2\epsilon$. There might be a last point on this part of $S_{a,b}$, or it might be a limit point not in the line segment.

10 Curves and Continuity and Dimensions

If a curve (C) is closed, and satisfies $C(x,\epsilon) \cup (C)$ is a finite set of points for all $x \in (C)$, except for a finite set of points (e.g. endpoints of the curve) $\forall \epsilon | \epsilon_0 > \epsilon > 0$, we can call the curve continuous. If the circle and curve intersect in two points (as with lines in the familiar Euclidean plane) for all but a few points (given small enough circles), we can try to define an ordering of points.

For example, pick a center for the circle b, give the circle radius $r < \epsilon_0$ and name the points of intersection with the curve a and c. If it is the case that $\rho(c,a) > r$ then a is farther from c than b, and the three points are ordered. Pick a smaller radius $r - \delta$ and call the points of intersection of this new circle a' and c'. $\delta_1 \equiv \rho(a,a') > \delta$ and $\delta_2 \equiv \rho(c,c') > \delta$. It is easy to see that the distance $\rho(c',a') > r - \delta_1 - \delta_2$. Unfortunately there aren't strong upper bounds on δ_1 and δ_2, so it isn't instantly obvious that $\rho(c',a') > r - \delta$. Were that true, we could extend ordering on the curve.

The obvious case to check is that in which all but a finite number of points on the curve have a single point in the intersection of arbitrary small circles with the curve. This isn't possible if, given $S_{a,b}$, $\forall \lambda \in [0, \rho(a,b)], \exists x \in S_{a,b}$ such that x is unique and $\rho(a,x) = \lambda$. This is easily seen by first finding a $c \in S_{a,b}$ and then a $d \in S_{a,c}$ such that $\rho(c,d) < \delta$. Then look at $S_{c,d}$. If the condition of every real distance having a point in the segment is true, then for the distance $\lambda = \rho(c,d)/2$ we can find a point f for which $\rho(c,f) = \lambda$. But then since by the definition of f being in $S_{c,d}$, we know that $\rho(d,f) = \lambda$. This gives us two points in the circle $C(f, \lambda) \cup S_{c,d}$, which contradicts the assumption the paragraph started with, since c and d are arbitrary (and therefore not restricted to a finite number of points).

At any rate, we can try to define a 1-dimensional curve as one which is continuous per the above definition (excepting a finite number of points) and for which each circle centered on a point on the curve intersects the curve in exactly two points (excepting that finite number of points on the curve). A two-dimensional surface would be one in which for almost all points on the surface, for a small enough radius of circle the intersection of the circle with the surface is a 1-dimensional curve. Of course the "almost all" is no longer merely "all but a finite number" anymore, our measure needs to be more sophisticated. One can make a ladder of dimensions this way, though the exceptional points get harder to define.

11 Other References

Taxicab Geometry by Eugene F. Krause is a nice elementary introduction that I just discovered (9-March-2004).