Effective Livetimes and their Applications

John Kelley and Gary Hill

August 3, 2007

1 Introduction

We consider the problem of determining the effective livetime of a sample of weighted events (such as in Monte Carlo simulations). We derive the expression for an effective livetime, then provide a few illustrative applications: optimization of cosmic ray simulation, and (more speculatively) estimating the statistical error on zero Monte Carlo events.

2 Formalism

We first present the idea of an effective number of events n_{eff}, as in [1]. For a set of n weighted events with observable x_i, each with weight w_i, the total number of weighted events T, given n simulated events, is

$$T = \sum_{i=1}^{n} w_i \ , \tag{1}$$

and the variance σ^2 is

$$\sigma^2 = \sum_{i=1}^{n} w_i^2 \ . \tag{2}$$

This leads naturally to the idea of an effective number of events n_{eff}, defined so that the fractional Poisson error on n_{eff} is the same as the weighted sample[1]:

$$\frac{n_{\text{eff}}^2}{(\sqrt{n_{\text{eff}}})^2} = \frac{T^2}{\sigma^2} \tag{3}$$

$$n_{\text{eff}} = \frac{T^2}{\sigma^2} = \frac{(\sum_{i=1}^{n} w_i)^2}{\sum_{i=1}^{n} w_i^2}. \tag{4}$$

[1] In ROOT, n_{eff} can be computed with TH1::GetEffectiveEntries().

2.1 Constant Event Weight

For a constant weight $w_i = w \; \forall \; i$, the effective number of events is just the unweighted number of Monte Carlo events:

$$n_{\mathrm{eff}} = \frac{(\sum_{i=1}^{n} w_i)^2}{\sum_{i=1}^{n} w_i^2} = \frac{n^2 w^2}{n \; w^2} = n \; . \tag{5}$$

Equivalently, one can view the weight w as the ratio between the weighted number of events T and n_{eff}:

$$\frac{T}{n_{\mathrm{eff}}} = \frac{n \; w}{n} = w \; . \tag{6}$$

Also note that in the case of constant weight w, the error σ on the weighted number of events T is just $w\sqrt{n_{\mathrm{eff}}}$:

$$\sigma = \sqrt{\sum_{i=1}^{n} w_i^2} = w\sqrt{n} = w\sqrt{n_{\mathrm{eff}}} \; . \tag{7}$$

One can view the weight w in terms of an effective livetime for the Monte Carlo sample, which perhaps results in a more intuitive feeling of how the errors are scaling. Specifically, if we are simulating a data sample (or integer-valued distribution) with livetime L, using Monte Carlo events with weight w, our effective livetime L_{eff} for the Monte Carlo sample is simply

$$L_{\mathrm{eff}} = \frac{L}{w} \; . \tag{8}$$

Viewed this way, w is the fraction L/L_{eff} by which we must scale the Monte Carlo distribution to result in one that has a Poisson variance. This also is equivalent to how we normally calculate simulation livetimes in the case of constant event weights, using the ratio of the data events (or weighted MC events normalized to data) to the number of simulated MC events:

$$L_{\mathrm{conv}} = L \; \frac{n}{N_{\mathrm{data}}} = L \; \frac{n}{T} = L \; \frac{n}{\sum_{i=1}^{n} w} = \frac{L}{w} = L_{\mathrm{eff}} \; . \tag{9}$$

2.2 Variable Event Weights

In many cases, Monte Carlo events have variable weights (such as in the case of spectral reweighting). We want to find the effective "average" weight $\tilde{w}$ that we can use to calculate an effective livetime. To do this, we generalize equation 6:

$$
\begin{aligned}
\tilde{w} &= \frac{T}{n_{\text{eff}}} \\
&= \frac{\sum w_i \ \sum w_i^2}{(\sum w_i)^2} \\
&= \frac{\sum w_i^2}{\sum w_i} \ .
\end{aligned} \tag{10}
$$

$\tilde{w}$ is the *contraharmonic mean* of the w_i, and for $w_i = w \ \forall \ i$, one can check that the above reduces to $\tilde{w} = w$. We also note that this definition of $\tilde{w}$ is equivalent to generalizing equation 7, so that

$$
\sigma = \tilde{w}\sqrt{n_{\text{eff}}} \ . \tag{11}
$$

In the language of livetimes, we are now in the position to define a effective livetime of a Monte Carlo subsample with variable event weights. Specifically, for a sample of events with weights w_i representing a data sample with livetime L, the effective livetime is

$$
L_{\text{eff}} = \frac{L}{\tilde{w}} = \frac{L \ \sum w_i}{\sum w_i^2} \ . \tag{12}
$$

Because $\tilde{w}$ is a function of the event subsample, one can define concepts like "the effective livetime in bin 10" or "the effective livetime above 100 GeV".

Note that while we derived this expression based on our definition of $\tilde{w}$, equation 12 is equivalent to another intuitive definition of L_{eff} based on n_{eff}:

$$
L\frac{n_{\text{eff}}}{T} = \frac{L \ \sum w_i}{\sum w_i^2} = L_{\text{eff}} \ . \tag{13}
$$

This is the variable-event-weight analogue to expression 9.

3 Application 1: Cosmic Ray Simulation

As a first application and sanity check of this definition, we apply the above formalism to the problem of cosmic ray simulation, in which one frequently simulates a harder spectrum than desired and then reweights to the original.

Specifically, we consider simulation of a power law spectrum $E^{-\gamma}$, using different spectral slopes $E^{-(\gamma+\Delta)}$. The event weights w_i for this case are

$$
w_i = \frac{\gamma - 1}{\gamma - 1 + \Delta} \ E_L^{-\Delta} \ E_i^{\Delta} \tag{14}
$$

where E_L is the low-energy bound for the simulation, and where the high-energy bound $E_H \gg E_L$ [2].

One can then generate a small sample of events with different Δ and compare the effective livetimes of events that trigger our detector (AMANDA-II, in this case), as shown in table 1. First, we note that the effective livetime is behaving as desired, and the effective livetime of high-energy events keeps rising as the spectrum gets harder. The effective livetime of low-energy events, however, starts to get worse as we oversample high energies and then reweight to a steep power law.

Because of the energy-dependent effective area of our detector, this leads to an optimal Δ to maximize the effective livetime of events at trigger level (in this case, $\Delta_{\text{best,L0}} = -0.6$). We can also find the energy range of events that survive to higher filter levels (say, level 3 of the 2005 filtering) and use this to estimate the best Δ for maximizing livetime at L3 (in this case, $\Delta_{\text{best,L3}} = -0.8$, because the energy peak at L3 is slightly higher than at L0).

Δ	$\gamma + \Delta$	Runtime (s)	Trig. events	L_{eff} (s)	L_{eff} (s) $E < 5$ TeV	L_{eff} (s) $E > 5$ TeV	L_{eff} (s) est. L3	L_{eff}/ runtime est. L3
0	-2.7	154	33	0.39	0.39	0.39	0.39	0.0025
-0.2	-2.5	176	44	0.59	0.46	0.67	0.59	0.0034
-0.4	-2.3	299	99	1.0	0.55	1.1	1.0	0.0034
-0.6	-2.1	508	188	1.3	0.57	1.9	1.3	0.0025
-0.8	-1.9	1454	361	1.2	0.54	2.4	1.5	0.0011
-1.0	-1.7	3745	875	1.2	0.50	3.5	1.5	0.0004

Table 1: Effective livetimes for cosmic ray MC samples with varying spectral slope. 50K events were simulated with dCORSIKA + SIBYLL, triggering AMANDA-II using Amasim.

Furthermore, one can take into account the variable (in some cases, nonlinear) simulation times for the different spectra (see the runtime column in table 1). Then one can choose the spectrum with the highest livetime to runtime ratio. For optimizing effective livetime to runtime at level 3, $\Delta_{\text{opt,L3}} = -0.4$. Note that simulation with $\Delta = -1$ is a factor of 6 times less efficient that using no slope change at all!

Of course, because the effective livetime depends on the event sample, the optimal Δ will depend on the specific filtering scenario for which one is optimizing. For high-energy filters, the harder slopes may be better, but keep in mind that this is only true if one has removed most of the low-energy events — otherwise their large weights will lower the livetime.

4 Application 2: The Error on Zero

Consider a Monte Carlo simulation of some binned distribution $f_i(x)$ of an event observable x (f is integer-valued in bins i), which falls off to zero at high x. A simulation of this distribution results will fall to zero at some $x > x_0$. We argue that the statistical error on this bin must depend on the number of simulated events n (unweighted) with $x < x_0$.

4.1 A Worst-case Scenario

Consider a worst-case scenario in which we have a single Monte Carlo event in bin j representing $f_i(x)$, that is, $n_j = 1$. Then the weight for this event in bin j is roughly the number of events f_j, if (as is most likely) the distribution peaks in bin j. The number of simulated events in bin $j+1$ is zero by construction, but the number of expected events f_{j+1} can be arbitrarily large depending on the distribution. Intuitively, we expect that the error on the simulated value $n_{j+1} = 0$ should be quite large, and ideally should cover the expected value f_{j+1}.

Specifically, let's suppose the expected number of events in bin j is 100, and the expected number of events in bin $j+1$ is 75, and that our single event is in bin j. So we'd expect the weight $w \approx 100$ if the distribution peaks around this value, and thus

$$\sigma \approx w\sqrt{n} = 100 \ . \tag{15}$$

and $T_j = 100 \pm 100$.

With an idea toward extending this to the $j+1$ bin, instead of using the error $\sqrt{n}$ above, we might also consider the Feldman-Cousins confidence interval [3] for $n_{obs} = 1$, which gives $\mu_{1\sigma} \in [0.37, 2.75]$, where μ is the "true" number of expected events (with infinite Monte Carlo). Then the weighted confidence interval is $w \cdot \mu \in [37, 275]$, or $T_j = 100^{+175}_{-63}$.

Now, in the $j+1$ bin, we have $n_{obs} = 0$, but now the event weight w_i is undefined. However, we're still considering in the case of a constant event weight, so we set $w = 100$ again. Now for $n_{obs} = 0$, the Feldman-Cousins confidence interval for the mean μ is $\mu_{1\sigma} \in [0, 1.29]$. Then the weighted confidence interval for this bin is $[0, 129]$, that is

$$T_{j+1} \approx 0^{+129}_{-0} \ . \tag{16}$$

Our hypothetical expected value for T_{j+1}, 75, lies within this interval, but we note this is because a) the weight w is a decent approximation for the expected value in bin j, and b) the expected value of bin $j+1$ is close to that of bin j. These are heuristic conditions for this approximation to remain meaningful.

Despite all the hand-waving, we are better off than before in that we have a handle on the statistical error on the simulated zero events in bin $j+1$, and we have an idea of how this depends on the event weighting.

Specifically, for constant event weight w and $n_{j+1} = 0$, we have

$$T_{j+1} \approx 0^{+w \cdot \mu_{CL}}_{-0} \ , \tag{17}$$

where $\mu_{1\sigma} = 1.29$ and $\mu_{90} = 2.44$.

4.2 Variable Event Weights

For variable event weights, we return to our "average" weight $\tilde{w}$ as defined in equation 10. We still have the problem, however, of the event weights being undefined in the zero bin. To approximate the weighting in this region, we construct a sequence $\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \ldots$ where

$$\tilde{w}_k = \frac{\sum_{bin=j-k-1}^{j} w_i^2}{\sum_{bin=j-k-1}^{j} w_i} \tag{18}$$

or, alternatively,

$$\tilde{w}_k = \frac{\sum_{bin=j-k-1} w_i^2}{\sum_{bin=j-k-1} w_i} \tag{19}$$

and bin $j+1$ is the first bin with zero simulated events. Then we construct an approximate limit (really, just an extrapolation)

$$\tilde{w}_0 = \lim_{k \to 0} \tilde{w}_k \ . \tag{20}$$

Then we use the estimated $\tilde{w}_0$ to construct the error on the zero bin $j+1$:

$$T_{j+1} \approx 0^{+\tilde{w}_0 \cdot \mu_{CL}}_{-0} \ . \tag{21}$$

From the viewpoint of effective livetimes, the sequence of $\tilde{w}_k$ extrapolated to $\tilde{w}_0$ can be seen as a sequence of effective livetimes L_k extrapolated to some estimated livetime representing the bin with zero events, L_0. The contents and error on that bin can equivalently be written as $0^{+(L/L_0) \cdot \mu_{CL}}_{-0}$.

Currently, we make no statement about the coverage of this modified confidence interval, as the accuracy of this approximation is dependent specifically on the weighting scheme and the shape of the observable distribution.

4.3 An Example

As an illustration of this error procedure, we consider the simulation of the number of optical modules hit (N_{ch}) in AMANDA-II by cosmic-ray muons, an energy-correlated observable. A plot of this distribution, simulated with a harder spectrum ($\Delta = -1.0$, so $\gamma = -1.7$), is shown in Figure 1. One notes that the high-energy bins have rather small errors (sub-Poissonian).

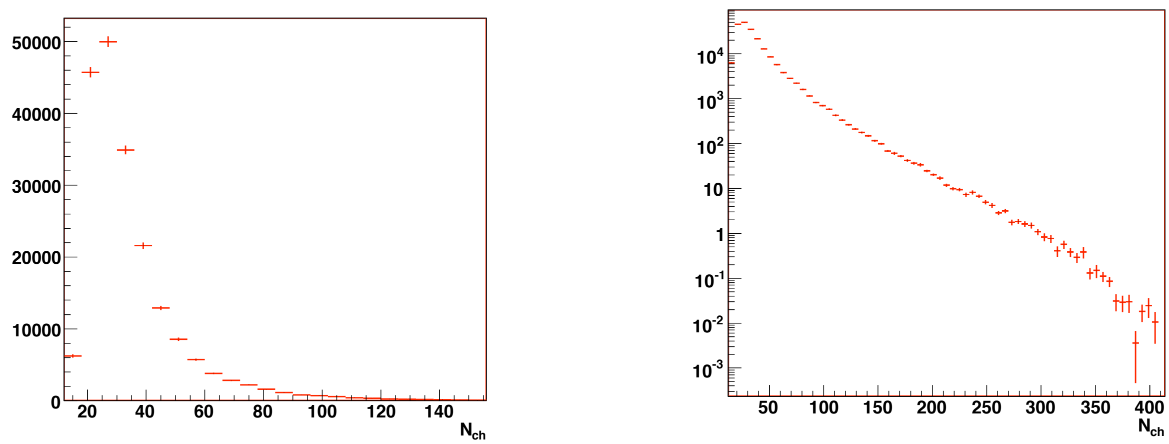

Figure 1: Number of optical modules hit, from simulation of atmospheric muons with $\Delta = -1.0$.

In Figure 2 one can see the effective weight $\tilde{w}$ calculated for each bin, and also running backward from the high-N_{ch} bin (as in 18). At low N_{ch}, the weight is significantly larger than 1, indicating the statistics are worse than Poissonian, while at high N_{ch}, the situation is reversed. We note that because the energy, and thus the weights w_i, are correlated with N_{ch}, $\tilde{w}$ varies smoothly across the distribution. Thus we can fairly easily extrapolate to $\tilde{w}_0$ for the bin ($414 < N_{ch} < 420$) — by eye, $\tilde{w}_0 \approx 0.006$, so

$$T_{414<N_{ch}<420} \approx 0^{+0.02}_{-0} \tag{22}$$

at the 90% confidence level. We note this error is quite reasonable given the values and errors of the final nonzero bins in figure 1.

4.4 A Caveat

The procedure to define the error on the zero bin with constant event weight w is always well-defined (by 17). It may be the case, however, that the sequence defined in equation 18 is not well-behaved. This can happen if the event weight w_i is not correlated with the observable chosen in the binning.

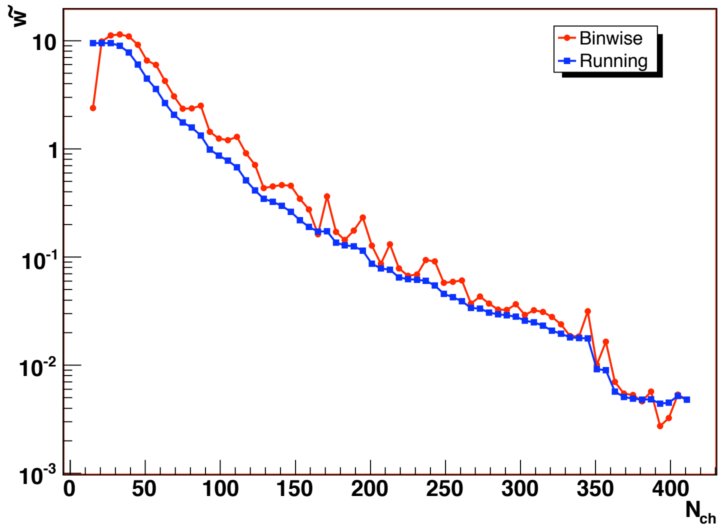

Figure 2: The effective weight $\tilde{w}$ calculated both for each bin of the N_{ch} distribution as well as the sample running back from the final bin.

In this case, it may not be possible to determine a limit or extrapolation of the $\tilde{w}_k$. One may at least, however, be able to estimate the order of magnitude of $\tilde{w}_0$.

References

[1] L. Lyons, *Statistics for nuclear and particle physicists*, Cambridge University Press (1986), 12-13.

[2] See `http://www.icecube.wisc.edu/~jkelley/simulation/dcors_weighting.pdf`.

[3] G. J. Feldman and R. D. Cousins. Phys. Rev. **D57**, 873 (1998).