Homework # 1 Solutions

1. (a) The magnitude of the Coulomb force between the protons in the molecule is

$$F_e = \frac{k_e q_1 q_2}{r^2} = \frac{k_e q_p^2}{r_s^2}, \tag{1}$$

with $q_p = 1.6 \times 10^{-19}$ C, and r_s is the separation of the protons in the molecule. The force is repulsive because it is between like sign charges.

(b) The gravitational force is

$$F_g = \frac{G m_1 m_2}{r^2} = \frac{G m_p^2}{r_s^2},$$

with $m_p = 1.67 \times 10^{-27}$ kg. Therefore,

$$\frac{F_e}{F_g} = \frac{k_e q_p^2}{G m_p^2} = 1.24 \times 10^{36}. \tag{2}$$

(c) For the gravitational force to equal the electrostatic Coulomb force between two objects of mass m and charge q,

$$\frac{k_e q^2}{r^2} = \frac{G m^2}{r^2},$$

such that

$$\frac{q}{m} = \sqrt{\frac{G}{k_e}} = 8.61 \times 10^{-11} \mathrm{C/kg}. \tag{3}$$

2. The condition for equilibrium is that the forces on the third bead are equal in magnitude and opposite in direction. Since both of the charges exerting force on the bead are positive, the forces have opposite directions when the bead is located between them at position x, with $0 < x < d$. Setting the magnitudes of the forces equal at x yields the condition

$$\frac{nq}{x^2} = \frac{q}{(d-x)^2}.$$

Since $0 < x < d$, this reduces to

$$\sqrt{n}(d-x) = x,$$

such that

$$x = \frac{\sqrt{n}}{1+\sqrt{n}} d. \tag{4}$$

If the bead has positive charge, it is in stable equilibrium; any small deviation of the bead from equilibrium will result in a net force acting to return the bead to its equilibrium position. If the bead has negative charge, it is in unstable equlibrium; any small deviation will result in a net force that will accelerate the bead away from its equilibrium position.

3. (a) The force $\vec{F}_{-Q}$ on charge $-Q$ is the vector sum of the forces on it from each of the $+q$ charges. To calculate $\vec{F}_{-Q}$, use the formula

$$\vec{F}_{-Q} = -k_e Q \sum_i \frac{q_i(\vec{r}_{-Q} - \vec{r}_{q_i})}{|\vec{r}_{-Q} - \vec{r}_{q_i}|^3},$$

where

$$\vec{r}_{-Q} = x\hat{x},\ \vec{r}_{q_1} = d/2\hat{y},\ \vec{r}_{q_1} = -d/2\hat{y},$$

such that

$$\vec{r}_{-Q} - \vec{r}_{q1,2} = x\hat{x} \mp (d/2)\hat{y},$$

and

$$|\vec{r}_{-Q} - \vec{r}_{q1,2}| = \sqrt{x^2 + (d/2)^2}.$$

The net force is

$$\vec{F}_{-Q} = -\frac{k_e Q q}{(x^2 + (d/2)^2)^{\frac{3}{2}}}\left(x\hat{x} - \frac{d}{2}\hat{y} + x\hat{x} + \frac{d}{2}\hat{y}\right) = -\frac{2k_e Q q x}{(x^2 + (d/2)^2)^{\frac{3}{2}}}\hat{x}. \tag{5}$$

Equivalently, note that the magnitude of the force on $-Q$ from each charge is $k_e|Q|q/(x^2 + (d/2)^2)$, such that the $\hat{x}$ and $\hat{y}$ components for the first charge are

$$F_x = -\frac{k_e|Q|q}{x^2 + (d/2)^2}\frac{x}{\sqrt{x^2 + (d/2)^2}},\ \ F_y = \frac{k_e|Q|q}{x^2 + (d/2)^2}\frac{d/2}{\sqrt{x^2 + (d/2)^2}},$$

and for the scond charge are

$$F_x = -\frac{k_e|Q|q}{x^2 + (d/2)^2}\frac{x}{\sqrt{x^2 + (d/2)^2}},\ \ F_y = -\frac{k_e|Q|q}{x^2 + (d/2)^2}\frac{d/2}{\sqrt{x^2 + (d/2)^2}}.$$

The vector sum is

$$\vec{F}_{-Q} = -\frac{2k_e Q q x}{(x^2 + (d/2)^2)^{\frac{3}{2}}}\hat{x},$$

as in Eq. (5) above. For $x \ll d/2$, we can neglect the x^2 term in the denominator, in which case $\vec{F}_{-Q}$ takes the Hooke's law form:

$$\vec{F}_{-Q} = -\frac{16k_e Q q}{d^3}x\hat{x} = -k_s x\hat{x}. \tag{6}$$

The motion is simple harmonic, with angular frequency

$$\omega = \sqrt{\frac{k_s}{m}} = 4\sqrt{\frac{k_e Q q}{md^3}}. \tag{7}$$

The period of the motion is

$$T = \frac{2\pi}{\omega} = \frac{\pi}{2}\sqrt{\frac{md^3}{k_e Q q}}. \tag{8}$$

(b) If the charge is released from rest at amplitude $x = a \ll d/2$, the speed at the midpoint can be calculated using energy conservation:

$$\frac{1}{2}k_s a^2 = \frac{1}{2}mv_m^2,$$

such that

$$|v_m| = \sqrt{\frac{k_{spring}}{m}}a = \omega a = 4a\sqrt{\frac{k_e Q q}{md^3}}. \tag{9}$$

4. Denote the distance between the charges as $l = 1.00$m, and set the origin of coordinates on charge q_1. Since $q_1 < 0$ and $q_2 > 0$, the contributions to the electric field from q_1 and q_2 are in opposite directions for $x < 0$ and for $x > l$, and in the same direction for $0 < x < l$. The x position at which the electric field is zero can be determined by equating the magnitudes of the two contributions to the field:

$$\frac{|q_1|}{x^2} = \frac{q_2}{(x-l)^2}. \tag{10}$$

Given that $q_2 > |q_1|$, this equation will be satisfied only for $x < l$. The solution with $x < 0$, which is where the field vanishes, is

$$x = -\frac{\sqrt{|q_1|}}{\sqrt{q_2} - \sqrt{|q_1|}}. \tag{11}$$

5. (a) The electric field at the origin due to the -3.00 nC charge and the charge q is given by

$$\vec{E} = -\frac{k_e q}{(0.3\text{ m})^2}\hat{x} - \frac{k_e(3.00 \times 10^{-9}\text{ C})}{(0.1\text{ m})^2}\hat{y}. \tag{12}$$

(b) The force on the 5.00 nC charge is

$$\vec{F} = -(5.00 \times 10^{-9}\text{ C})\left(\frac{k_e q}{(0.3\text{ m})^2}\hat{x} - \frac{k_e(3.00 \times 10^{-9}\text{ C})}{(0.1\text{ m})^2}\hat{y}\right). \tag{13}$$

6. Use the formula

$$\vec{E}(\vec{r}) = k_e \int \frac{dq(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}')|^3}.$$

Here $\vec{r} = 0$, $\vec{r}' = x'\hat{x}$, and $dq = \lambda_0 dx'$, such that

$$\vec{E} = -\hat{x}\int_{x_0}^{\infty} \frac{k_e \lambda_0 dx'}{x'^2} = -\hat{x}\left(-\frac{k_e\lambda_0}{x'}\right)_{x_0}^{\infty} = -\hat{x}\frac{k_e\lambda_0}{x_0}. \tag{14}$$

7. The acceleration of a particle with charge q and mass m in an electric field $\vec{E}$ is

$$\vec{a} = \frac{q}{m}\vec{E}.$$

Since this is a 1-d problem, the velocity of the particle in time t is given by

$$v = v_0 + at = at = \frac{qEt}{m}, \tag{15}$$

since $v_0 = 0$. The speed of the electron and proton then are

$$v_e = \frac{(1.6 \times 10^{-19}\text{ C})Et}{9.11 \times 10^{-31}\text{ kg}}, \qquad v_p = \frac{(1.6 \times 10^{-19}\text{ C})Et}{1.67 \times 10^{-27}\text{ kg}}.$$

The electron has a negative charge, so it moves in a direction opposite to the electric field. The proton has a positive charge, so it moves in the direction of the electric field.