Homework # 4 Solutions

1. The parallel-plate capacitor is charged then disconnected from the battery before the plate separation d is changed. Therefore, the charge is fixed. Since the capacitance is

$$C_0 = \frac{\epsilon_0 A}{d},$$

when the plate separation changes to $d \to nd$, the new capacitance is $C = C_0/n$. The energy is

$$U = \frac{Q^2}{2C} = \frac{Q^2 n}{2C_0} = nU_0,$$

where U_0 is the original stored energy in the capacitor.

2. (a) We have a parallel-plate capacitor which is charged and disconnected from the battery. The charge on the plates both before and after the immersion is

$$Q = C_0 \Delta V_0 = \frac{\epsilon_0 A}{d} \Delta V_0.$$

(b) The capacitance after the immersion is $C = \kappa C_0$, where $\kappa = 80$ for distilled water. The potential difference is

$$\Delta V = \frac{Q}{C} = \frac{Q}{\kappa C_0} = \frac{\Delta V_0}{\kappa}.$$

(c) The change in energy of the capacitor is

$$\Delta U = U_{\text{water}} - U_{\text{air}} = \frac{Q^2}{2C} - \frac{Q^2}{2C_0} = \frac{Q^2}{2C_0}\left(\frac{1}{\kappa} - 1\right).$$

3. (a) Given $q(t) = 4t^3 + 5t + 6$, the instantaneous current is

$$I(t) = \frac{dq}{dt} = 12t^2 + 5.$$

(b) The current density is $J = I/A$.

4. The resistance is given by

$$R = \frac{\rho l}{A},$$

where ρ is the resistivity, l is the length, and A is the cross-sectional area (here $A = \pi r^2$). For two wires of different resistivities and the same length to have the same resistance, their radii must obey the following condition:

$$\frac{r_1}{r_2} = \sqrt{\frac{\rho_1}{\rho_2}}.$$

5. (a) The current in the resistor R is given by the terminal voltage V divided by R:

$$I = \frac{V}{R}.$$

(b) The emf of the battery is given by the formula

$$\mathcal{E} = I(R + r),$$

where r is the internal resistance.

6. Call the current in the circuit when the switch is opened I_0. Kirchhoff's loop rule yields

$$\mathcal{E} - I_0(R_1 + R_2 + R_3) = 0.$$

When the switch is closed to position 1 (call the current in this case I_1), the loop rule gives

$$\mathcal{E} - I_1\left(R_1 + \frac{R_2}{2} + R_3\right) = 0.$$

Note that the two resistors of value R_2 are in parallel here, with net resistance $R_2/2$. When the switch is closed to position 2 (current I_2), the loop rule gives

$$\mathcal{E} - I_2(R_1 + R_2) = 0.$$

There are three equations to solve for the three unknown resistances:

$$\begin{aligned} R_1 + R_2 + R_3 &= \frac{\mathcal{E}}{I_0} \\ R_1 + \frac{R_2}{2} + R_3 &= \frac{\mathcal{E}}{I_1} \\ R_1 + R_2 &= \frac{\mathcal{E}}{I_2} \end{aligned}$$

The solution is:

$$R_1 = -2\frac{\mathcal{E}}{I_0} + 2\frac{\mathcal{E}}{I_1} + \frac{\mathcal{E}}{I_2}; \quad R_2 = 2\frac{\mathcal{E}}{I_0} - 2\frac{\mathcal{E}}{I_1}; \quad R_3 = \frac{\mathcal{E}}{I_0} - \frac{\mathcal{E}}{I_2}.$$

7. (a) The time constant is $\tau = RC$.

(b) The maximum charge on the capacitor is $Q = \mathcal{E}C$: recall for a charging capacitor

$$Q(t) = \mathcal{E}C\left(1 - e^{-t/RC}\right).$$

(c) The instantaneous current is

$$I(t) = \frac{dQ}{dt} = \frac{\mathcal{E}}{R}e^{-t/RC}.$$

8. This is a two-loop circuit, with two independent currents. I defined them to be I_1 pointing from point a to point b, and I_2 pointing from point f to point c. By the junction rule, the current pointing from point e to point d is $I_3 = -(I_1 + I_2)$. (Recall you can choose the independent currents in any way you wish, as long as you apply the junction rule properly.)

(a) To obtain the currents, we use Kirchhoff's loop rule twice. I chose to traverse clockwise around loop $abcf$:

$$\mathcal{E}_1 - \mathcal{E}_2 + I_2 R_2 - I_1 R_1 = 0,$$

and to traverse clockwise around loop $fcde$:

$$\mathcal{E}_2 + I_3 R_3 - \mathcal{E}_3 - I_2 R_2 = 0.$$

The currents are:

$$\begin{aligned} I_1 &= \frac{\mathcal{E}_1(R_2 + R_3) - \mathcal{E}_2 R_3 - \mathcal{E}_3 R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3} \\ I_2 &= \frac{\mathcal{E}_2(R_1 + R_3) - \mathcal{E}_1 R_3 - \mathcal{E}_3 R_1}{R_1 R_2 + R_1 R_3 + R_2 R_3} \\ I_3 &= \frac{\mathcal{E}_3(R_1 + R_2) - \mathcal{E}_1 R_2 - \mathcal{E}_2 R_1}{R_1 R_2 + R_1 R_3 + R_2 R_3}. \end{aligned}$$

The current through R_1 is I_1, the current through R_2 is $-I_2$ (*i.e.*, the current in that branch flows from point c to point f, opposite to the direction I assigned for it initially), and the current through R_3 is I_3.

(b) The potential difference between points c and f is: $|-\mathcal{E}_2 + I_2 R_2|$ (given my above definition of I_2). Point c is at higher potential.