Homework # 6 Solutions

1. The magnetic field at the center of the loop is the superposition of the magnetic field due to the long wire and the magnetic field at the center of the loop:

$$B = \frac{\mu_0 I}{2\pi R} + \frac{\mu_0 I}{2R} = \frac{\mu_0 I}{2R}\left(1 + \frac{1}{\pi}\right).$$

2. At point P, the contributions to the magnetic field arise from the two $\pi/3$ arcs with radii a and b (the magnetic field at P due to the sides of the loop is zero, since there $d\vec{l}$ is parallel to the point $\vec{r}$, such that $Id\vec{l} \times \vec{r} = 0$). Since the currents are oppositely directed, the magnetic field is the difference between the two:

$$B = \frac{\mu_0 I}{4\pi a}\frac{\pi}{3} - \frac{\mu_0 I}{4\pi b}\frac{\pi}{3} = \frac{\mu_0 I}{12}\left(\frac{1}{a} - \frac{1}{b}\right).$$

Of the two arcs, the counterclockwise current (in the arc of radius a) gives the larger contribution to the magnetic field, and thus the direction of the field is given by the right-hand rule to be out of the plane of the page.

3. (a) Denote the separation of the wires by d. The magnitude of the magnetic field created by wire 1 at the location of wire 2 is

$$B_1 = \frac{\mu_0 I_1}{2\pi d},$$

and the direction is out of the plane of the page.

(b) The force per unit length exerted on I_2 by I_1 is

$$\frac{F}{l} = I_2 B_1 = \frac{\mu_0 I_1 I_2}{2\pi d},$$

directed downward (in the negative y direction).

(c) The magnitude of the magnetic field created by wire 2 at the location of wire 1 is

$$B_2 = \frac{\mu_0 I_2}{2\pi d},$$

and the direction is into of the plane of the page.

(d) The force per unit length exerted on I_1 by I_2 is

$$\frac{F}{l} = I_1 B_2 = \frac{\mu_0 I_1 I_2}{2\pi d},$$

directed upward (in the positive y direction).

4. By Ampere's Law, the magnetic field at point a (radius a) is given by

$$\oint \vec{B} \cdot d\vec{s} = B 2\pi a = \mu_0 I_{enclosed} = \mu_0 I_1,$$

such that

$$B = \frac{mu_0 I_1}{2\pi a}.$$

The field circulates in a counterclockwise direction around I_1; at point a it is directed upward.

At point b, the current enclosed is $I_1 - I_2$, so by the same logic

$$\oint \vec{B} \cdot d\vec{s} = B 2\pi b = \mu_0 I_{enclosed} = \mu_0 (I_1 - I_2),$$

and hence

$$B = \frac{mu_0 (I_1 - I_2)}{2\pi b}.$$

If $I_2 > I_1$, the B field will be clockwise in direction, and therefore will point downward at point b.

5. The magnetic field of the solenoid is $B = \mu_0 n I_s$, where n is the number of turns per unit length and I_s is the current in the solenoid. The loop is perpendicular to the field direction. Looking at the loop through one end of the solenoid (with current I_s circulating in the clockwise direction), the magnetic field of the solenoid is directed into the plane defined by the loop. Since the field of the solenoid is uniform, the force on each side of the loop is given by

$$F = I_l l B = \mu_0 n I_l I_s l,$$

in which I_l is the current in the loop and l denotes each edge of the square loop. Since the loop current is also flowing in the clockwise direction, the direction of the magnetic force is outward, directed away from the center in the plane of the loop.

The net torque on the loop is zero. This can be seen by inspection of the directions of the forces (which are equal in magnitude) on each side of the square. Alternatively, recall that a current loop in a magnetic field will feel a torque such that its magnetic dipole moment aligns with the magnetic field. Here the magnetic moment vector of the loop and the magnetic field of the solenoid are in the same direction, so there is zero net torque.

6. (a) The magnetic flux through surface S_1 is

$$\vec{B} \cdot \vec{A} = -BA\cos\theta = -B\pi R^2 \cos\theta.$$

(The minus sign is due to the fact that the area vector is defined to be the outward pointing normal to S_1, which is in the opposite direction of the parallel component of the magnetic field. Since this is slightly ambiguous, please keep in mind that the answer without the minus sign would net full credit were this an exam question.)

(b) By Gauss's law for magnetism, which states that the net flux through any *closed* surface is zero, the flux through S_2 must be

$$B\pi R^2 \cos\theta,$$

i.e., equal and opposite to the flux through S_1.

7. (a) The electric field in between the capacitor plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{q}{\pi R^2 \epsilon_0},$$

where R is the radius of the capacitor plates. Since q is changing as a function of time, with

$$I = \frac{dq}{dt},$$

the time rate of change of the electric field is

$$\frac{dE}{dt} = \frac{dq}{dt}\frac{1}{\pi R^2 \epsilon_0} = \frac{I}{\pi R^2 \epsilon_0}.$$

(b) The source for the magnetic field between the capacitor plates is the displacement current,

$$I_d = \epsilon_0 \frac{d\Phi_e}{dt} = \epsilon_0 \pi R^2 \frac{dE}{dt} = I$$

(from part (a) above). To obtain the magnetic field at radius r, use Ampere-Maxwell's Law:

$$\oint \vec{B} \cdot d\vec{s} = B 2\pi r = \mu_0 (I + I_d)_{enclosed} = (I_d)_{enclosed} = I_d \frac{r^2}{R^2} = I \frac{r^2}{R^2}$$

such that

$$B = \frac{\mu_0 I r}{2\pi R^2}.$$

Here we used the fact that the displacement current is effectively uniformly distributed between the plates of the capacitor, such that the enclosed displacement current is equal to the total current times the ratio of areas $\pi r^2/(\pi R^2)$. (Therefore, the full amount of the displacement current is only enclosed for a path the size of the radius of the capacitor plate or larger. Note that for such large paths one has to also be concerned with edge effects of the capacitor's E field. However, we will not concern ourselves with such issues in Physics 202.)

8. Since the thickness of the toroid is negligible compared to its mean radius r_m, the magnetic field of the iron-core toroid is given by

$$B \approx \frac{\mu N I}{2\pi r_m}.$$

(Note that for magnetic materials, effectively one replaces μ_0 with the magnetic permeability μ in the formula for the magnetic field. The current is then

$$I = \frac{B 2\pi r_m}{\mu N}.$$

9. By Faraday's Law,

$$|\mathcal{E}| = IR = \frac{d\Phi_B}{dt} = \left(A \frac{\Delta B}{\Delta t} \right),$$

where A is the cross-sectional area, I is the induced current, and R is the resistance of the loop. The induced current is thus

$$I = \frac{|\mathcal{E}|}{R} = \frac{A}{R}\frac{\Delta B}{\Delta t}.$$