Ph 801 — Exercise 4

1. Derive the maximum energy transfer in a 2-body scattering of a particle of mass m and momentum **p** on a target particle (eg. an atomic electron) that is at rest of mass m_e.

Suggested solution

Let us consider an incident particle of mass m and momentum **p** and a target particle of mass m_e (eg atomic electron) at rest.
Conservation of energy before and after the scattering:

$$\sqrt{p^2c^2 + m^2c^4} + m_ec^2 = \sqrt{p''^2c^2 + m^2c^4} + E_k + m_ec^2 \quad \text{where}$$

$$E_k + m_ec^2 = \sqrt{p'^2c^2 + m_e^2c^4}$$

Momentum conservation

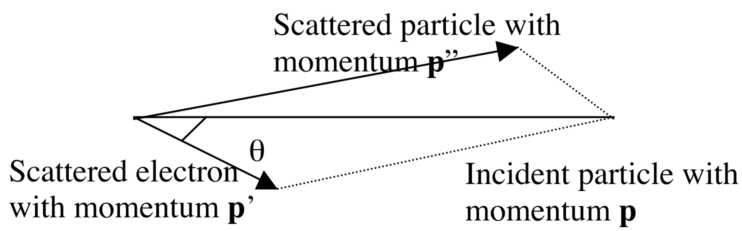
p" = p – p'

$p''^2 = p^2 + p'^2 - 2pp' \cos\theta$

From $E_k + m_ec^2 = \sqrt{p'^2c^2 + m_e^2c^4} \Rightarrow p'^2 = \dfrac{(E_k + m_ec^2)^2 - m_e^2c^4}{c^2}$

and

$p''^2 = p^2 + p'^2 - 2pp' \cos\theta$



(*) $$p''^2 = p^2 + \frac{(E_k + m_ec^2)^2 - m_e^2c^4}{c^2} - 2p\cos\theta\sqrt{\frac{(E_k + m_ec^2)^2 - m_e^2c^4}{c^2}}$$

But $\sqrt{p^2c^2 + m^2c^4} + m_ec^2 = \sqrt{p''^2c^2 + m^2c^4} + E_k + m_ec^2$

$$\Rightarrow \sqrt{p^2c^2 + m^2c^4} = \sqrt{p''^2c^2 + m^2c^4} + E_k \Rightarrow p^2c^2 + m^2c^4 + E_k^2 - 2E_k\sqrt{p^2c^2 + m^2c^4} = p''^2c^2 + m^2c^4$$

$$p^2 + E_k^2/c^2 - 2E_k\sqrt{p^2c^2 + m^2c^4}\,/c^2 = p''^2$$

Substituting in (*)

$$p^2 + E_k^2/c^2 - 2E_k\sqrt{p^2c^2 + m^2c^4}\,/c^2 = p^2 + \frac{(E_k + m_ec^2)^2 - m_e^2c^4}{c^2} - 2p\cos\theta\sqrt{\frac{(E_k + m_ec^2)^2 - m_e^2c^4}{c^2}}$$

$$\Rightarrow E_k^2/c^2 - 2E_k\sqrt{p^2c^2 + m^2c^4}\,/c^2 = \frac{(E_k + m_ec^2)^2 - m_e^2c^4}{c^2} - 2p\cos\theta\sqrt{\frac{(E_k + m_ec^2)^2 - m_e^2c^4}{c^2}}$$

$$\Rightarrow E_k^2 - 2E_k\sqrt{p^2c^2 + m^2c^4} = (E_k + m_ec^2)^2 - m_e^2c^4 - 2pc\cos\theta\sqrt{(E_k + m_ec^2)^2 - m_e^2c^4}$$

$$\Rightarrow E_k^2 - 2E_k\sqrt{p^2c^2 + m^2c^4} = E_k^2 + m_e^2c^4 + 2E_km_ec^2 - m_e^2c^4 - 2pc\cos\theta\sqrt{(E_k + m_ec^2)^2 - m_e^2c^4}$$

$$\Rightarrow -2E_k\sqrt{p^2c^2 + m^2c^4} = 2E_km_ec^2 - 2pc\cos\theta\sqrt{E_k^2 + 2E_km_ec^2}$$

$$\Rightarrow m_e c^2 + \sqrt{p^2c^2 + m^2c^4} = pc\cos\theta\sqrt{\frac{E_k^2 + 2E_k m_e c^2}{E_k^2}}$$

$$\Rightarrow \left(m_e c^2 + \sqrt{p^2c^2 + m^2c^4}\right)^2 = p^2c^2\cos^2\theta\,\frac{E_k^2 + 2E_k m_e c^2}{E_k^2}$$

$$\Rightarrow \frac{\left(m_e c^2 + \sqrt{p^2c^2 + m^2c^4}\right)^2}{p^2c^2\cos^2\theta} = 1 + \frac{2m_e c^2}{E_k}$$

$$\Rightarrow E_k = \frac{2m_e c^2}{\left[\frac{\left(m_e c^2 + \sqrt{p^2c^2 + m^2c^4}\right)^2}{p^2c^2\cos^2\theta} - 1\right]} = \frac{2m_e c^4 p^2\cos^2\theta}{\left(m_e c^2 + \sqrt{p^2c^2 + m^2c^4}\right)^2 - p^2c^2\cos^2\theta}$$

The maximum energy transfer is obtained for $\cos\theta = 1$ that is $\theta = 0$ (head-on collision)

$$\Rightarrow E_{k,\max} = \frac{2m_e c^4 p^2}{\left(m_e c^2 + \sqrt{p^2c^2 + m^2c^4}\right)^2 - p^2c^2} = \frac{2m_e c^4 p^2}{m_e^2 c^4 + 2m_e c^2\sqrt{p^2c^2 + m^2c^4} + m^2c^4} = \frac{p^2c^2}{\frac{1}{2}m_e c^2 + \sqrt{p^2c^2 + m^2c^4} + \frac{1}{2}{m^2}/{m_e}\,c^2}$$

Since the incoming particle energy is

$$E_i = \sqrt{p^2c^2 + m^2c^4} = m\gamma c^2 \Rightarrow p^2c^2 = m^2\gamma^2c^4 - m^2c^4$$

$$\Rightarrow E_{k,\max} = \frac{m^2\gamma^2c^4 - m^2c^4}{\frac{1}{2}m_e c^2 + m\gamma c^2 + \frac{1}{2}{m^2}/{m_e}\,c^2} = \frac{2m^2c^2\beta^2\gamma^2}{m_e + 2m\gamma + {m^2}/{m_e}} = \frac{2m_e c^2\beta^2\gamma^2}{1 + 2\frac{m_e}{m}\gamma + \left(\frac{m_e}{m}\right)^2}$$

For m>>me $\Rightarrow E_{k,\max} \approx 2m_e c^2\beta^2\gamma^2$