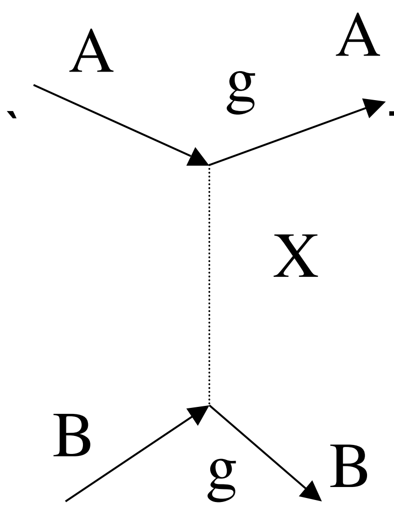
Range of a force

Elastic scattering of A and B via the exchange of X. In the rest frame of A the lower vertex represents the virtual process $A(M_A c^2, \mathbf{0}) \to A(E_A, \mathbf{p}) + X(E_X, -\mathbf{p})$



$$E_X = \sqrt{p^2c^2 + M_X^2 c^4} \quad E_A = \sqrt{p^2c^2 + M_A^2 c^4}$$

The energy difference between the initial and final states is:

$$\Delta E = E_X + E_A - M_A c^2 \to pc \text{ for p} \to \infty$$
$$\to M_X c^2 \text{ for p} \to 0$$

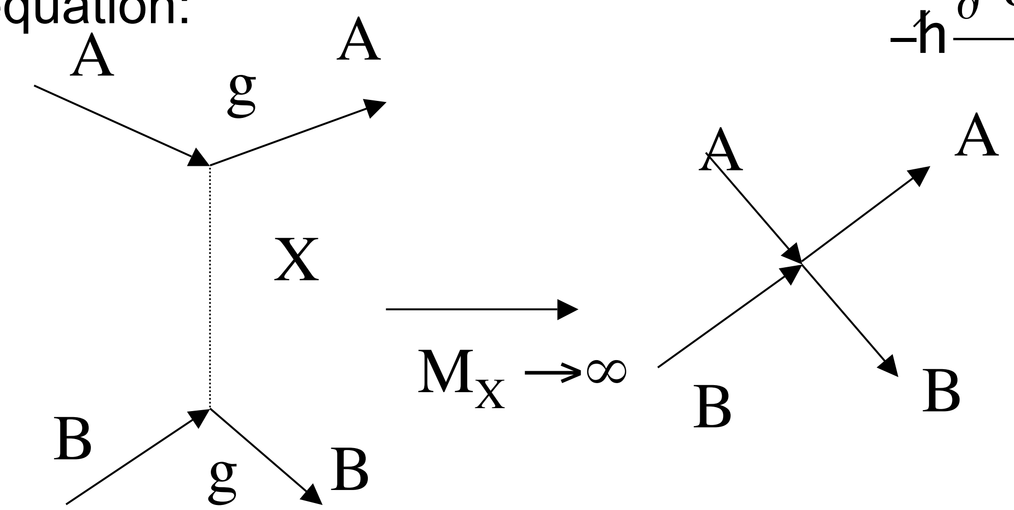
So the energy is violated ($\Delta E \geq M_X c^2$ for all p) for a time $\tau \approx \hbar/\Delta E$ and the maximum distance over which X can propagate is called the range of the interaction $R = c\tau \approx \hbar/(M_X c)$

Eg for em interactions it is infinite since the photon is massless. For weak interactions propagated by massive bosons (M_W = 80.6 GeV/c^2 and M_Z = 91.2 GeV/c^2 $R_W = 2\ 10^{-3}$ fm (1 fm = 10^{-15} m))

Particle exchange

Considering the exchange of a spin-0 boson X obeying the Klein-Gordon equation:

$$-\hbar \frac{\partial^2 \Psi(\mathbf{x},t)}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \Psi(\mathbf{x},t) + M_X^2 c^4 \Psi(\mathbf{x},t)$$



For static solutions the equation is

(*) $$\nabla^2 \Psi(\mathbf{x}) = \frac{M_X^2 c^2}{\hbar^2} \Psi(\mathbf{x})$$

and the solution is the static potential. For $M_X = 0$ the equation is that obeyed by the electrostatic potential and for a charge e^- interacting with a charge e^+ at the origin the solution is $V(r) = -e\Psi(r) = -\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$ where r = |**x**| and the general solution of (*) is the Yukawa potential

$$\Psi(r) = \frac{g}{4\pi}\frac{e^{-r/R}}{r}$$

with $R = \hbar/(M_X c)$. The coupling constant that characterizes the strength of the interaction is $\alpha_X = \frac{g^2}{4\pi\hbar c}$

Yukawa firstly introduced in 1935 the idea of forces exchanged by massive particles

Rates and Cross section

In a typical scattering experiment an ideally monoenergetic beam of particles is directed on a target and the rates of production of various particles is measured
The rate per unit area (flux) at which beam particles cross a small surface placed in the beam at rest respect to the target and perpendicular to the beam direction is $J = n_b v_i$, with n_b = density of beam particles and v_i = their velocity in the rest frame of the particle. The rate at which a reaction r occurs is $W_r = JN\sigma_r$
N = number of particles in the target illuminated by the beam
Where σ_r = cross-section for reaction r and $\sigma = \sum_r \sigma_r$ is the total cross section
L = JN is the luminosity.
The rate per target particle $J\sigma_r$ at which the reaction occurs is equal to the rate at which beam particles would hit a surface area σ_r placed in the beam at rest with respect to the target and perpendicular to the beam direction. The area is unchanged by a Lorentz transformation in the beam direction, it is the same in the laboratory or the CM frames.
The total reaction rate is $W = JN\sigma$. Since every reaction removes a particle from the incoming beam, σ determines how the beam intensity is reduced after crossing a target.

Total Cross section

The reduction in the intensity of the beam of cross-sectional area A crossing a segment of the target of thickness dx at a distance x within it is equal to the interaction rate in the segment:

$$AdJ(x) = -dW$$

Where dJ is the change in the flux J(x) between x and x+dx.

The rate is also given by $$dW = J(x)\sigma n_t A dx$$

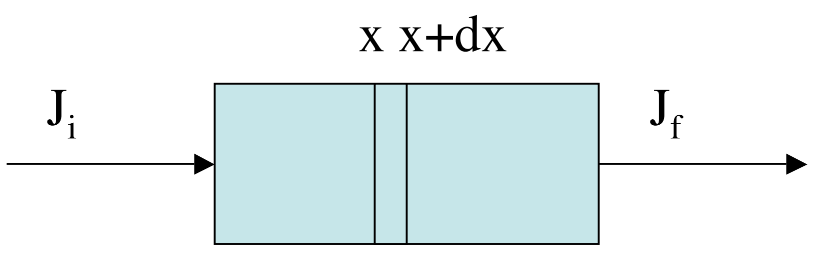
Where $N = n_t\, dV = n_t\, A\, dx$ and n_t = density of particles in the target

Hence $$dJ(x) = -n_t \sigma J(x) dx$$

Integrating and considering that the initial incident flux is $J(0) = J_i$:

$$J(x) = J_i \exp\left[-x / l_c\right]$$

l_c = collision length = $1/(n_t\sigma)$



Differential Cross section

Let's consider the angular distribution of the particles produced in a scattering reaction. The scattering angle θ is the angle between the beam direction and the particle direction in an elastic scattering (eg $\pi^- + p \to \pi^- + p$ and the production angle in an inelastic one $\pi^- + p \to K^0 + \Lambda$)

The angular distribution of the chosen particle (π^- or K^0) produced in a 2-body reaction r is described by the differential cross section defined by:

$$dW_r \equiv JN \frac{d\sigma_r(\theta,\phi)}{d\Omega} d\Omega$$

Where dW_r is the measured rate for the particle to be emitted into an element of solid angle $d\Omega = d\cos\theta d\phi$ in the direction (θ,ϕ). The relation between the reaction cross section and the differential one is $\sigma_r = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \frac{d\sigma_r(\theta,\phi)}{d\Omega}$

Most times experiments are done using unpolarized beams (no spin dependence and cylindrical spin around beam direction) and unpolarized cross sections are independent on ϕ

Scattering Amplitude

We can relate the cross-section to a probability of transition from an initial state i to a final one f. Let's consider a single beam particle interacting on a single target particle and let's confine the system in a cube of side L and volume $V = L^3$. The incident flux is $J = n_b v_i = v_i/V$ (since N=1). Hence the measured rate is:

$$dW_r \equiv JN \frac{d\sigma_r(\theta,\phi)}{d\Omega} d\Omega = \frac{\mathrm{v}_i}{\mathrm{V}} \frac{d\sigma_r(\theta,\phi)}{d\Omega} d\Omega$$

Let's consider the scattering of a non-relativistic particle by a potential V(**x**).

The initial state is described by the wavefunction: $\Psi_i = \frac{1}{\sqrt{V}} \exp[i\mathbf{q_i} \cdot \mathbf{x}]$

And the final state $\Psi_f = \frac{1}{\sqrt{V}} \exp[i\mathbf{q_f} \cdot \mathbf{x}]$

Where the final momentum $\mathbf{q}_f$ lies in a small solid angle dΩ in the direction (θ,φ). In non-relativistic quantum mechanics the transition rate is:

$$dW_r = 2\pi \left| \int d^3\mathbf{x}\, \Psi_f^* V(\mathbf{x}) \Psi_i \right|^2 \rho(E_f)$$

with $\rho(E_f)$ = density of final states

And defining the scattering amplitude associated to the potential (Fourier transform of the potential) M_{if}:

$$dW_r = \frac{2\pi}{V} \rho(E_f) \left| M_{if} \right|^2$$

Scattering Amplitude

Defining the momentum transfer **q**= $\mathbf{q}_f$ - $\mathbf{q}_i$ the scattering amplitude associated to the potential (Fourier transform of the potential) is:

$$M_{if} = g_0 \int d^3\mathbf{x}\, V(\mathbf{x}) \exp[i\mathbf{q}\cdot\mathbf{x}]$$

where g_0 is the coupling strength of the particle to the potential

For a central potential, in spherical coordinates V(**x**) = V(r), **q·r** = qrcosθ and dVol = $r^2 dr d\phi \sin\theta d\theta$ and R = ħ/mc

$$M_{if} = g_0 \int V(\mathbf{r}) e^{i(\mathbf{q}\cdot\mathbf{r})} d\mathbf{r} = 2\pi g_0 \int_0^\infty dr r^2 \frac{g}{4\pi r} e^{-r/R} e^{-iqr\cos\theta} \int_0^\pi \sin\theta d\theta = -g_0 \int_0^\infty dr e^{-mr} r \frac{g}{2} \int_1^{-1} e^{-iqry} dy =$$

$$= g g_0 \int_0^\infty dr e^{-mr} \frac{e^{iqr} - e^{-iqr}}{2iq} = g g_0 \left[\frac{e^{-(m-iq)r}}{-2iq(m-iq)} - \frac{e^{-(m+iq)r}}{-2iq(m+iq)} \right]_0^\infty =$$

$$= g g_0 \left[\frac{1}{-2iq(m-iq)} + \frac{1}{2iq(m+iq)} \right] = g g_0 \left[\frac{-(m-iq)+(m+iq)}{2iq\left[m^2-(iq)^2\right]} \right] = \frac{g g_0}{|\mathbf{q}|^2 + m^2}$$

$1/(q^2+m^2)$ propagator

The cross section is the product of $|f|^2$ times a phase space factor divided by the incident flux

Rates and Cross section

The density of the final states can be found considering that the possible values of the momentum (for the particle in $V=L^3$) are $q_{x,y,z}=(2\pi/L)n_{x,y,z}$ with $n_{x,y,z}$ integers and the number of states with momenta in the momentum space volume $d^3\mathbf{q} = q^2 dq d\Omega$ (momenta pointing into the solid angle $d\Omega$ with magnitude between q, q+dq) is $\rho(q)dq = \left(\frac{L}{2\pi}\right)^3 d^3\mathbf{q} = \frac{V}{8\pi^3} q^2 dq d\Omega$

$$\rho(E)dE = \rho(q)\frac{dq}{dE}dE \qquad E = \frac{1}{2}mv^2 \Rightarrow dE = mvdv = vdq \qquad \rho(E_f) = \frac{\rho(q_f)}{v_f} = \frac{V}{8\pi^3}\frac{q_f^2}{v_f}d\Omega$$

$$\Rightarrow \quad dW_r = \frac{2\pi}{V}\rho(E_f)\left|M_{if}\right|^2 = \frac{2\pi}{V}\frac{V}{8\pi^3}\frac{q_f^2}{v_f}d\Omega\left|M_{if}\right|^2 = \frac{V}{4\pi^2}\frac{q_f^2}{v_f}d\Omega\left|M_{if}\right|^2 = \frac{\mathbf{v}_i}{\mathbf{V}}\frac{d\sigma_r(\theta,\phi)}{d\Omega}d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2}\frac{q_f^2}{v_f v_i}d\Omega\left|M_{if}\right|^2$$

$$dW_r \equiv JN\frac{d\sigma_r(\theta,\phi)}{d\Omega}d\Omega = \frac{\mathrm{v}_i}{\mathrm{V}}\frac{d\sigma_r(\theta,\phi)}{d\Omega}d\Omega$$

The cross section provides the probability of occurrence of an interaction
Typically measured in units of barn: 1 barn = 10^{-24} cm^2 = 10^{-28} m^2
The interaction length is $\lambda_I = 1/(n_A\,\sigma_{tot}) = A/(N_A\rho\,\sigma_{tot})$ where
$n_A = N_A\rho/A$ = reciprocal of atomic volume

The 4 known interactions

Electromagnetic: electrons are bound in atoms by it. Its strength is governed by the fine structure constant (as all the other coupling constants, it ‘runs’ - increases -with the energy eg at 100 GeV α_{em} ~1/128).

$$\alpha_{em} = \frac{e^2}{4\pi\hbar c} \approx 1/137$$

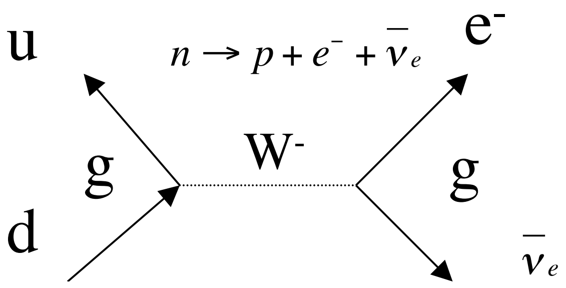
The spin 1 bosons carrying the force are photons.

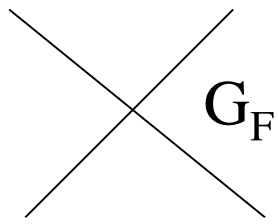
The weak force is responsible of the β-decay of nuclei and its coupling constant is: $g_{weak} = \frac{e}{\sin\theta_W}$ with θ_W = Weimberg angle and $\sin^2\theta_W$ ~0.23

The fact that these 2 constants are of the same order of magnitude is related to the fact that they are unified in the SM (electroweak interactions) and the corresponding quantum theory is called QED (quantum electrodynamics).

Before it was known that W,Z (spin 1 bosons) existed, Fermi described weak interactions (β-decay) as a pointlike interactions with the Fermi constant

$$n \to p + e^- + \bar{\nu}_e$$





$$\frac{G_F}{\sqrt{2}} = \frac{g_{weak}^2}{8M_W^2} \approx 1.166 \cdot 10^{-5}\,GeV^{-2}$$

The 4 known interactions

Hadrons (e.g. the n and p) are made of quarks that exist only in their bound states and are bound together by the strong interactions. They bounds nucleons in nuclei. The coupling constant runs faster (decreases) with energy than em int. At few GeV: $\alpha_s = \frac{g_s}{4\pi} \approx 0.3$

Quarks interact via the exchange of mediating bosons called gluons. While in QED there are two kind of charges (+ and -), in the theory of interquark forces (QCD = quantum cromodynamics) there are 6 kinds of strong charges called colors (quarks can carry 3 kind of colors - red, green, blue - and antiquarks 3 kinds of anticolors). Gluons are the carriers of the strong force.

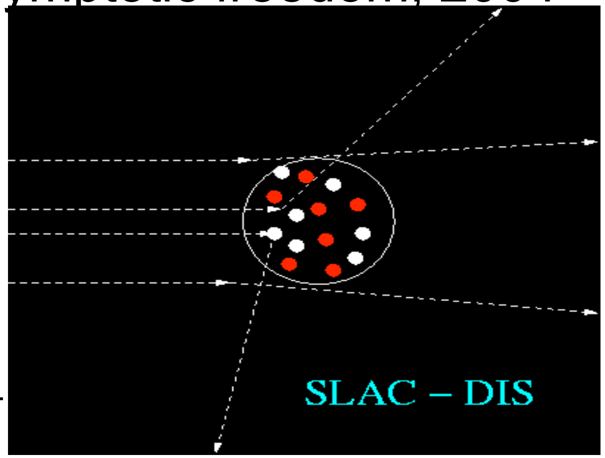
The potential between 2 quarks can be described by:

$$V_s \approx -\frac{4}{3}\frac{\alpha_s}{r} + kr$$

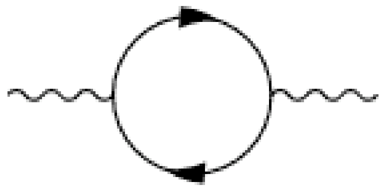
the 1st term dominates at small distances and arises from single gluon exchange (similar to Coulomb potential $-\alpha/r$). The 2nd is associated to confinement of quarks at large distances. Because of this term,attempts to free a quark from a hadron results in production of $Q\overline{Q}$ pairs. The lines of force of the color field are pulled together by a strong gluon-gluon interaction forming a flux tube or string. Pulling it out, the stored energy kr reaches a point that a couple $Q\overline{Q}$ can be created.

The 4 known interactions

In QCD the interactions between quarks are represented by the running const. $\alpha_s(q^2)$: for $q^2 \to\infty$ $\alpha_s(q^2) \to 0$ quarks behave as free (asymptotic freedom, 2004 Nobel prize to Gross, Politzer, Wilczek) eg in parton model interactions on single quarks are considered. The opposite behavior respect to α is due to the fact that gluons carry the color charge while the photon is neutral.



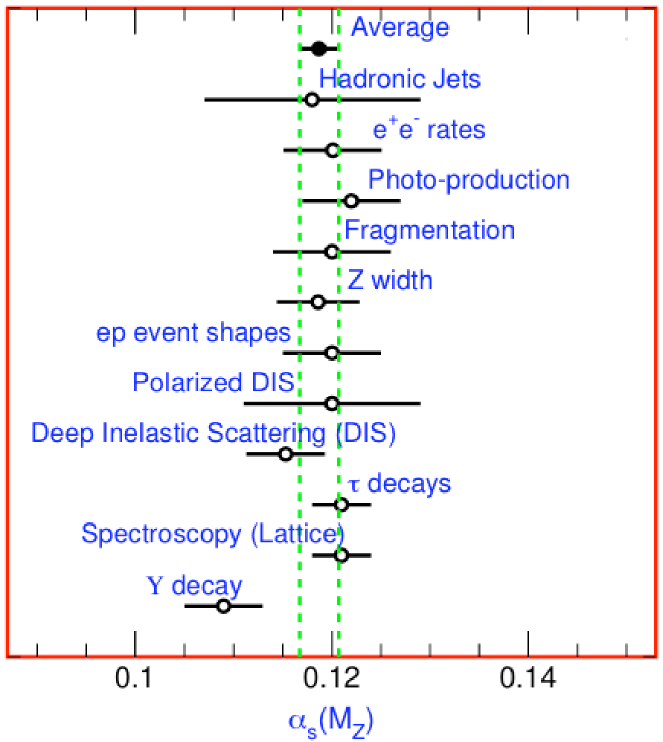
An electron can emit a photon that annihilates in a e^--e^+ pair, so the electron is surrounded by a cloud of charges (the positive ones are attracted closer to the electron) and his charge is partially screened. This is why α increases with increasing energy (decreasing distance) because the closer one approaches the electron the larger the charge one measures. A similar screening effect occurs for q-anti-q pairs. However gluons carry color and an anticolor charge, and the effect of couples of virtual gluons is a resulting antiscreening that leads to the asymptotic freedom.

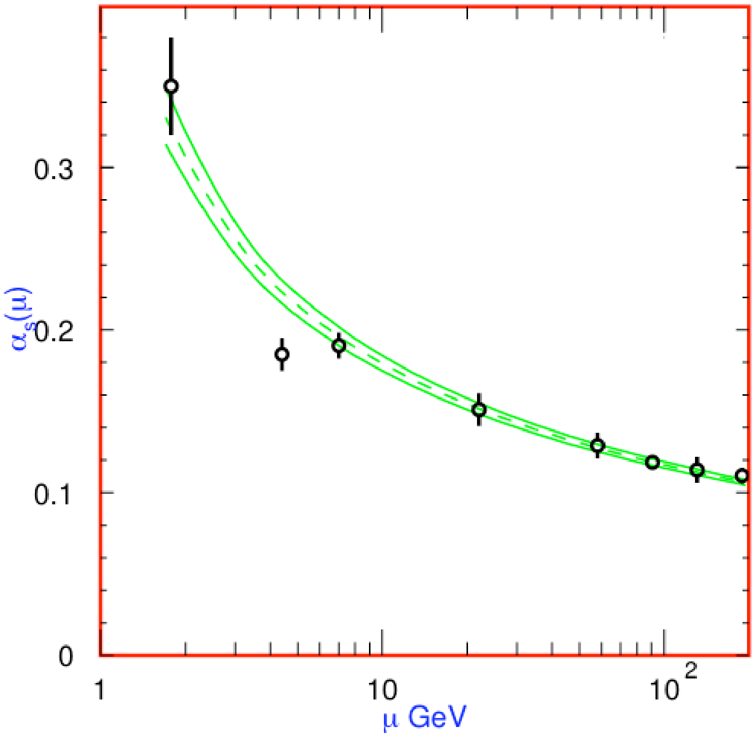


In addition there exists gravity. Considering the proton mass $KM^2/(4\pi\hbar c) \sim 10^{-40}$. The graviton has not been discovered yet.

Notice: hadrons are color neutral states. Gluons have zero mass hence the force between quarks must be of long range. This does not imply that forces between hadrons are also long range, because hadrons have zero color charges overall. The forces between colorless hadrons are the residues of forces between their quark constituents, and cancel when hadrons are far apart.

How good is QCD?





Operators and conservation laws

In quantum mechanics (Emmy Noether 1918) **any invariance in equations under a continuous change of variables is related to a conservation law.**
The existence of conserved quantum numbers can be traced back to an invariance of the theory under some set of transformations.
Translational invariance ↔ linear momentum
Rotational invariance ↔ angular momentum
Invariance under gauge transformations ↔ electric charge
Given an em field described by the potentials (ϕ,**A**) the electric and magnetic fields are unchanged under $\phi \rightarrow \phi' = \phi + \frac{\partial f}{\partial t}$ $\mathbf{A} \rightarrow \mathbf{A'} = \mathbf{A} - \nabla f$
Where f is an arbitrary scalar function.
Other types of transformations (acting on internal degrees of freedom) lead to conserved quantum numbers such as baryon and lepton numbers but these Conservation laws may not be absolute: eg for barion number defined as B = [N(q)-N($\bar{q}$)]/3 (for barions made of 3 quarks = 1)
Exercise 2:

- What is violated?

$$\mu^- \rightarrow e^-$$
$$t \rightarrow \bar{s} + \bar{b}$$
$$b \rightarrow s + \bar{s} + d + e^- + \bar{\nu}_e$$

The building blocks of matter and interacting forces

FERMIONS
matter constituents
spin = 1/2, 3/2, 5/2, ...

Leptons spin = 1/2

Flavor	Mass GeV/c^2	Electric charge
ν_e electron neutrino	$<1\times10^{-8}$	0
e electron	0.000511	−1
ν_μ muon neutrino	<0.0002	0
μ muon	0.106	−1
ν_τ tau neutrino	<0.02	0
τ tau	1.7771	−1

Quarks spin = 1/2

Flavor	Approx. Mass GeV/c^2	Electric charge
u up	0.003	2/3
d down	0.006	−1/3
c charm	1.3	2/3
s strange	0.1	−1/3
t top	175	2/3
b bottom	4.3	−1/3

BOSONS
force carriers
spin = 0, 1, 2, ...

Unified Electroweak spin = 1

Name	Mass GeV/c^2	Electric charge
γ photon	0	0
W^-	80.4	−1
W^+	80.4	+1
Z^0	91.187	0

Strong (color) spin = 1

Name	Mass GeV/c^2	Electric charge
g gluon	0	0

PROPERTIES OF THE INTERACTIONS

Property \ Interaction	Gravitational	Weak (Electroweak)	Electromagnetic (Electroweak)	Strong: Fundamental	Strong: Residual
Acts on:	Mass – Energy	Flavor	Electric Charge	Color Charge	See Residual Strong Interaction Note
Particles experiencing:	All	Quarks, Leptons	Electrically charged	Quarks, Gluons	Hadrons
Particles mediating:	Graviton (not yet observed)	W^+ W^- Z^0	γ	Gluons	Mesons
Strength relative to electromag for two u quarks at: 10^{-18} m	10^{-41}	0.8	1	25	Not applicable to quarks
Strength relative to electromag for two u quarks at: 3×10^{-17} m	10^{-41}	10^{-4}	1	60	
for two protons in nucleus	10^{-36}	10^{-7}	1	Not applicable to hadrons	20

Leptons

They are spin 1/2 fermions without strong interactions

They occur in pairs called generations (also the corresponding antileptons exist): each generation has a conserved quantum number called lepton number

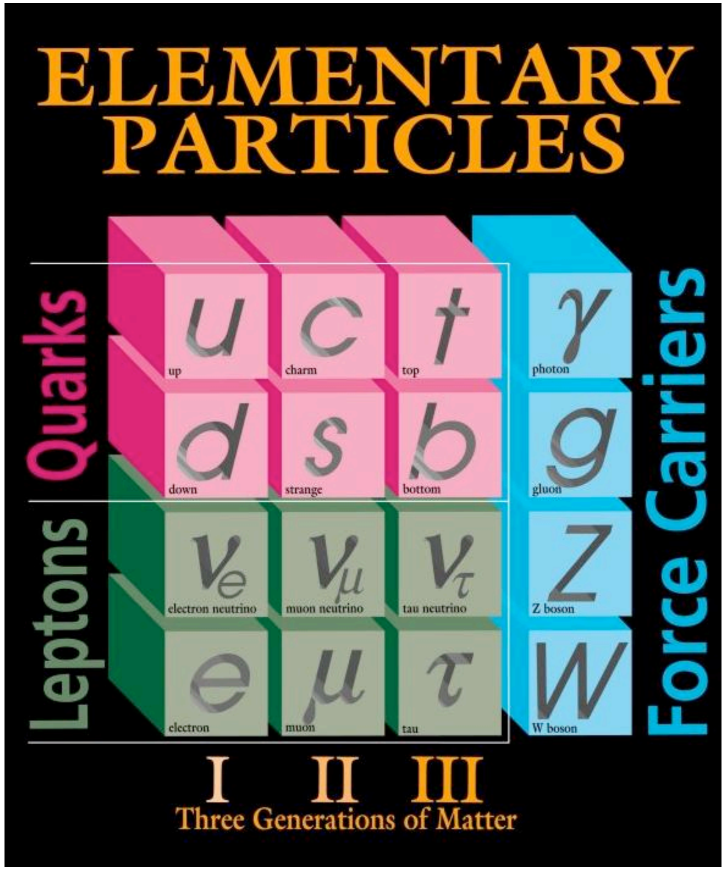
$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}$$

$L_e = N(e^-) - N(e^+) + N(\nu_e) - N(\bar{\nu}_e)$

so that $L_e = 1$ for electrons and ν_e

-1 for positrons and $\bar{\nu}_e$

0 for other particles



Similarly

$L_\mu = N(\mu^-) - N(\mu^+) + N(\nu_\mu) - N(\nu_\mu)$

$L_\tau = N(\tau^-) - N(\tau^+) + N(\nu_\tau) - N(\nu_\tau)$

Neutrino oscillations violate lepton number conservation

Lepton masses

Exercise 2

e	$m = 0.51099892 \pm 0.00000004$ MeV	ν_e	$m < 3$ eV	Trizium β-decay
μ	$m = 105.658369 \pm 0.000009$ MeV	ν_μ	$m < 0.19$ MeV, CL = 90%	from π decay
τ	$m = 1776.99^{+0.29}_{-0.26}$ MeV	ν_τ	$m < 18.2$ MeV, CL = 95%	from $\tau \to 5\pi^{\pm}\pi^0\nu_\tau$

Notice that these are upper limits on flavor eigenstates that are related to mass eigenstates by $|\nu_\ell\rangle = \sum_i U_{\ell i} |\nu_i\rangle$

Eg. In the β-decay of trizium $\bar{\nu}_e$ are produced and $m^{2(\mathrm{eff})}_{\nu_e} = \sum_i |U_{ei}|^2 m^2_{\nu_i}$

is constrained. This provides independently on the mixing parameters $|U_{ei}|^2$ a limit on the minimum value $m^2_{\nu_{\min}} \leq m^{2(\mathrm{eff})}_{\nu_e}$. but only knowing all Δm^2_{ij} and $|U_{ei}|^2$ allows to determine the individual masses

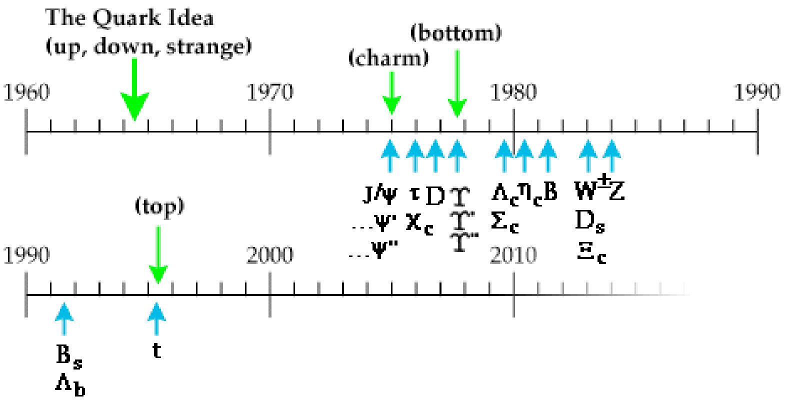
$$m^2_{\nu_j} = m^{2(eff)}_{\nu_e} - \sum_i |U_{ei}|^2 \Delta m^2_{ij}$$

Number $N = 2.994 \pm 0.012$ (Standard Model fits to LEP data)

Number $N = 2.92 \pm 0.07$ (Direct measurement of invisible Z width)

Quark masses

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad \begin{pmatrix} c \\ s \end{pmatrix} \quad \begin{pmatrix} t \\ b \end{pmatrix}$$



u $m = 1.5$ to 4 MeV Charge $= \frac{2}{3}\,e$ $\quad I_z = +\frac{1}{2}$

d $m = 4$ to 8 MeV Charge $= -\frac{1}{3}\,e$ $\quad I_z = -\frac{1}{2}$

s $m = 80$ to 130 MeV Charge $= -\frac{1}{3}\,e$ Strangeness $= -1$

c $m = 1.15$ to 1.35 GeV Charge $= \frac{2}{3}\,e$ Charm $= +1$

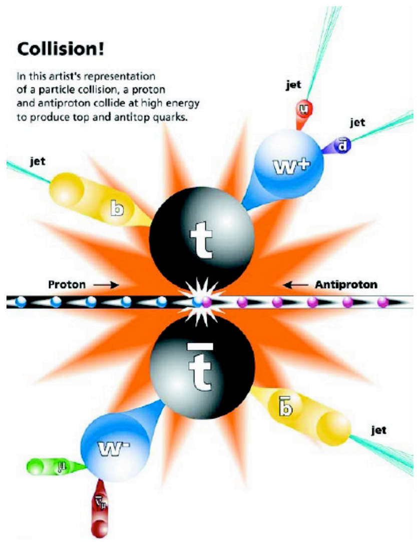
b $m = 4.1$ to 4.4 GeV Charge $= -\frac{1}{3}\,e$ Bottom $= -1$

t $m = 174.3 \pm 5.1$ GeV (direct observation of top events)

$m = 178.1^{+10.4}_{-8.3}$ GeV (Standard Model electroweak fit)

Discovered in 1995 at Tevatron

Quarks exist inside hadrons not as free states, therefore masses cannot be measured directly but can be inferred through their influence on hadronic properties They have fractional charges -1/3 and 2/3. In 1964 Zweig and Gell-Mann independently noted that hadrons can be considered as bound states of these quarks



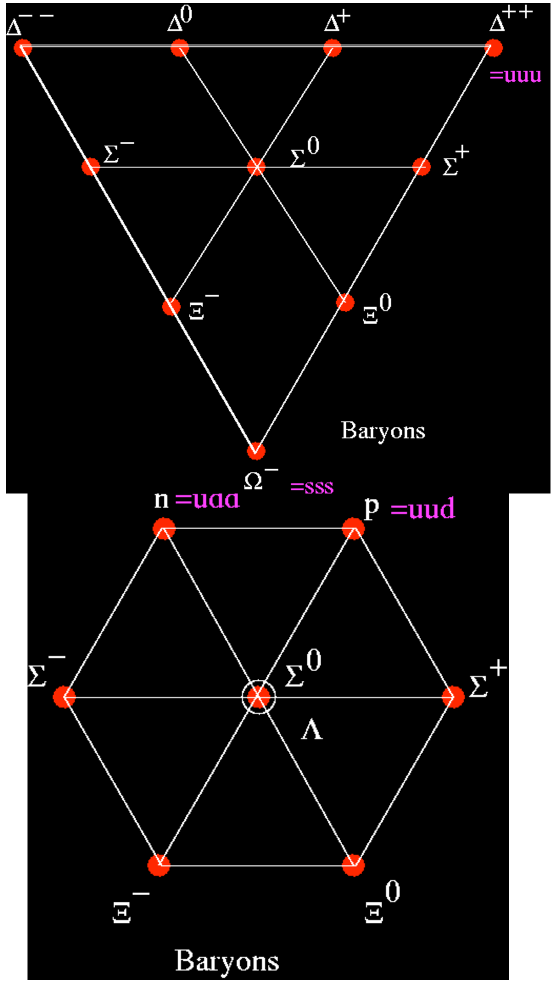
Hadrons: barions and mesons

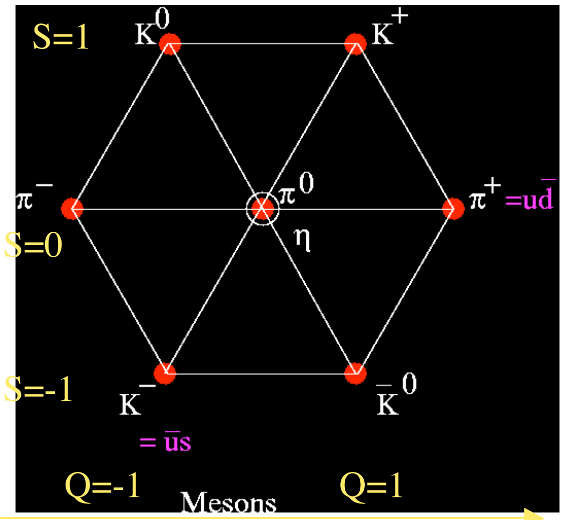
Baryons = qqq Mesons = $q\bar{q}$ color neutral states

Baryon decuplet $J^P = 3/2^+$ and octet $J^P = 1/2^+$

Meson octet $J^P = 0^-$

(J is the angular momentum, P the intrinsic parity
= parity of the particle at rest)





The eightfold way

P is a unitary operator such that P $\Psi(\mathbf{r}) \rightarrow \Psi(-\mathbf{r})$ conserved in strong and em interactions. Fermion and antifermion must have opposite parity (from Dirac equation). Leptons have parity 1 and also quarks. Parity of mesons is

$P_M = P_q P_{\bar{q}}(-1)^L = (-1)^{L+1}$ L=orbital angular momentum

$(-1)^L$ is the parity of the space part of a wave function

Hadron quantum numbers

Each strongly interacting particle (hadron) is characterized by its mass and several quantum numbers: spin-parity J^P (eg proton $1/2^+$ has spin 1/2 and parity +1) and internal quantum numbers, eg electric charge, baryon number B, Strangeness S, charm C, beauty B, truth T which are conserved in strong and em interactions. These are additive quantum numbers that can be found by adding together the contributions from quarks and anti-quarks. T= 0 for all known hadrons since till now no hadron has been found containing a t quark. In 1932 Heisenberg suggested that n and p are different charge substates of the same particle, the nucleon to which a quantum number, the isospin, is ascribed. For the nucleon I = 1/2 and the 2 substates have I_3 = +1/2 (proton) and -1/2 (neutron). Strong interactions conserve the isospin (they are invariant under rotations of the length of the isospin vector). It is found that

$Q/e = I_3 + B/2+S/2$

Eg the pion is an isospin triplet with I = 1 and 3 substates π^+, π^0, π^-

(K^0,K^+) is a doublet with S = + 1 and (K^-,K^0) with S = - 1 and I=1/2